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Non-Semisimple Symmetry TFTs

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Introduction

In recent years a lot of effort was spent on understanding and studying various notions of **generalised symmetries** including higher-form and non-invertible symmetries. These new kinds of symmetries show up in a variety of systems, ranging from the Ising model, over topological phases of matter, to string theory.

generalised symmetries in QFTs no longer fit into the conventional mathematical framework in terms of groups, instead the correct framework is provided by **(higher) category theory** and **Topological Field Theories (TFTs)**.

Symmetries and topological defects

A **defect** in a QFT is a lower-dimensional region of spacetime which behaves differently then its surroundings. The resulting theory is "defective" in the sense that two regions of spacetime disjoint by a defect could have vastly different physical properties. A defect is called **topological** if its precise location does not matter.

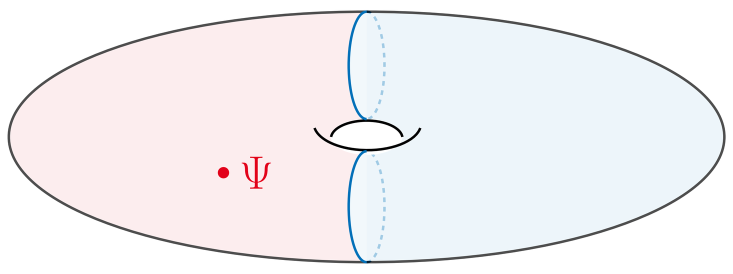


Figure 1. Theory on a "world sheet" $\mathcal{S}$ with 1-dimensional defects and field insertion Ψ.

Topological defects generalise the notion of an ordinary symmetry. To see this take an element g of the symmetry group G. We can interpret g as a topological defect of codimension 1 across which the value of field operators jumps by the action of g:

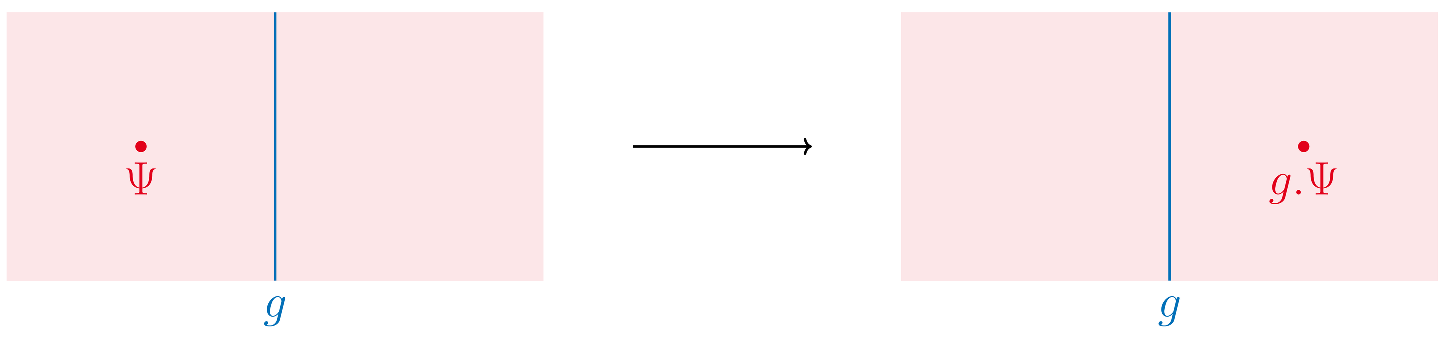


Figure 2. Moving a field Ψ past the defect corresponding to g changes the field by the action of g to $g.\Psi$.

More precisely, the topological defect corresponds to an operator U_g acting on the Hilbert space $\mathcal{H}$ of our theory. The family of operators $(U_g)_{g\in G}$ together with the Hilbert space $\mathcal{H}$ form a **representation** of the abstract symmetry group G. Analogously one should think of the topological defects in a given QFT as a representation of the actual symmetry structure.

Symmetry TFTs

Going from a given representation to the abstract symmetry group gives us a deeper understanding of the symmetry itself, beyond its specific representation. A **symmetry TFT (symTFT)** is the abstract entity governing all topological defects of the QFT we are interested in. The idea is to **encode all topological defects** of a d-dimensional QFT $\mathcal{T}$ in a $(d+1)$-dimensional TFT $\mathcal{Z}$ with a topological "Dirichlet" **boundary condition** $\mathcal{B}$. The symTFT $\mathcal{Z}$ acts on the QFT $\mathcal{T}$ via dimensional reduction along an interval:

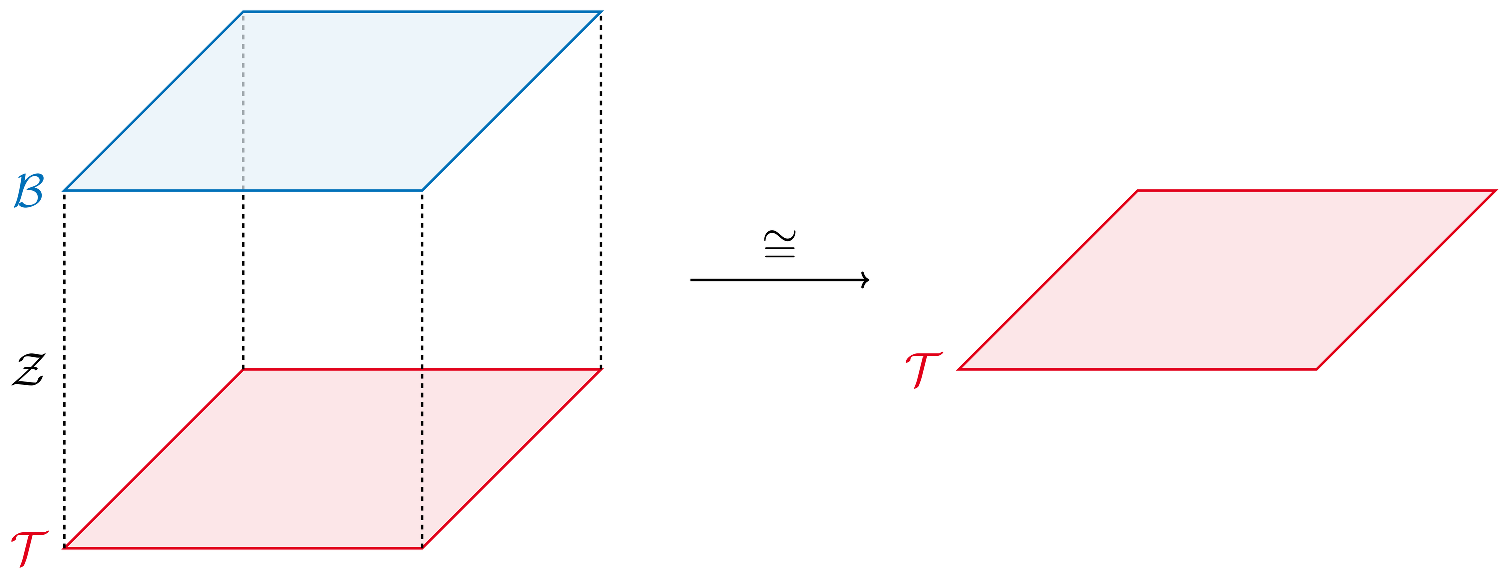


Figure 3. Action of a symmetry TFT $\mathcal{Z}$ on the theory $\mathcal{T}$ via the boundary condition $\mathcal{B}$.

TFTs themselves are studied in pure mathematics using (higher) **category theory**.

2d CFTs and 3d TFTs

These ideas allow us to rigorously study 2-dimensional Conformal Field Theories. To this end a 2d CFT will mean the following mathematical objects for us:

- A **vertex operator algebra** (VOA) $\mathcal{V}$ (formalizing the algebra of chiral operators)
- A **modular functor** $\mathrm{Bl}_{\mathcal{V}}$ (assignment of conformal block spaces, compatible with cutting/gluing of surfaces)
- The **field content** (Specific VOA modules)
- **Correlators** (elements in the block spaces satisfying certain conditions)

In the **rational/semisimple** setting one can reconstruct the whole CFT from the VOA $\mathcal{V}$ and one extra datum via the FRS-construction [2]. Let us illustrate this for the one point correlator $\langle\Psi\rangle_{\mathcal{S}}$ on the world sheet $\mathcal{S}$ in Fig. 1.

For any rational VOA $\mathcal{V}$ there is a 3d TFT $\mathcal{Z}_{\mathcal{V}}$ with $\mathrm{Rep}(\mathcal{V})$ as its category of line operators. The **state spaces** $\mathcal{Z}_{\mathcal{V}}(\Sigma)$ are conjecturally **isomorphic** to the chiral **conformal blocks** $\mathrm{Bl}_{\mathcal{V}}(\Sigma)$.

Under this conjecture we can obtain $\langle\Psi\rangle_{\mathcal{S}}$ by the following procedure:

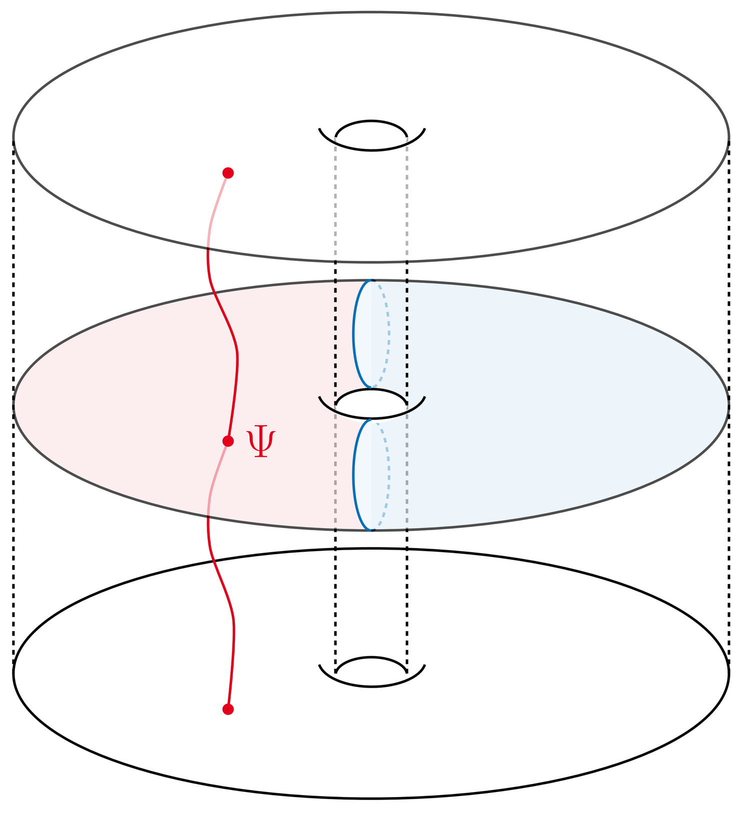


Figure 4. Cylinder over the world sheet $\mathcal{S}$.

Consider the (punctured) surface Σ underlying $\mathcal{S}$. Next take the cylinder $\Sigma\times[-1,1]$ and embed $\mathcal{S}$ as a **surface defect** at $\Sigma\times\{0\}$ as in Fig. 4. Applying $\mathcal{Z}_{\mathcal{V}}$ to the cylinder produces a vector in the state space $\mathcal{Z}_{\mathcal{V}}(\Sigma\sqcup-\Sigma)$.

It can be shown that this vector automatically satisfies the conditions of a correlator! In particular we can explicitly **compute correlators** by working with **diagrams** in $\mathrm{Rep}(\mathcal{V})$!

To arrive at the symTFT picture, fold the cylinder in half and note that the surface defect becomes a **boundary condition** for the **doubled** TFT $\mathcal{Z}_{\mathcal{V}}\otimes\overline{\mathcal{Z}_{\mathcal{V}}}$.

Rationality is a strong finiteness assumption, can we weaken this to study more classes of CFTs? In [1] a (partially defined) 3d TFT with *non-semisimple* category of line operators was constructed, can this be used as a **non-semisimple symmetry TFT** for finite logarithmic CFTs?

Why are non-semisimple theories interesting?

From a physics perspective:

- Applications in statistical physics, e.g. critical dense polymers.
- Wess-Zumino-Witten models with supergroup target are often non-semisimple.
- Twists of supersymmetric QFTs are usually non-semisimple, even derived.

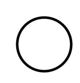
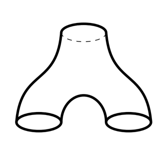
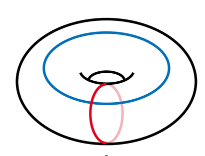
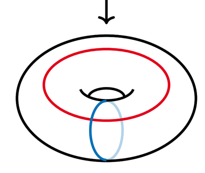
From a mathematics perspective:

- Most 2d TFTs are non-semisimple.
- Can we understand other non-semisimple CFT constructions from the 3d perspective?
- Stronger topological invariants.
- Topological interpretation of algebraic structures.
- Step towards derived TFTs.

Main result

Theorem[3]
For any modular tensor category $\mathcal{C}$, the 3d TFT of [1] with $\mathcal{C}$ as category of line operators induces a chiral modular functor

$$\mathrm{Bl}^{\chi}_{\mathcal{C}}\colon \mathrm{Bord}^{\chi}_{2+\epsilon,2,1}\to\mathcal{P}\mathrm{rof}^{\mathcal{L}\mathrm{ex}}_{\Bbbk}.$$

$\mathrm{Bord}^{\chi}_{2+\epsilon,2,1}$	$\mathcal{P}\mathrm{rof}^{\mathcal{L}\mathrm{ex}}_{\Bbbk}$	CFT interpretation
	finite linear category $\mathcal{C}$	Representation category of VOA
	$\mathrm{Hom}_{\mathcal{C}}(U\otimes V,W)$	Space of three point correlators on sphere
	$\mathrm{Hom}_{\mathcal{C}}(\mathbb{1},\mathcal{L})$	Modular S-transformation of VOA characters
↓	↓ S	$\chi_i(-\frac{1}{\tau})=\sum_j S_{ij}\chi_j(\tau)+\ldots$
	$\mathrm{Hom}_{\mathcal{C}}(\mathbb{1},\mathcal{L})$	

Future work

In order to fully exploit the FRS construction we need non-semisimple TFTs which include **surface defects**.

- What is the right algebraic input? ⇒ **pivotal module categories**
- How do we construct the TFTs? ⇒ **internal state sum/gauge line defects**

In contrast to the semisimple setting, we expect not all surface defects to come from the gauging of line defects. **How can we get the others?**

References

[1] M. De Renzi, A. M. Gainutdinov, N. Geer, B. Patureau-Mirand, and I. Runkel. "3-dimensional TQFTs from non-semisimple modular categories". In: *Selecta Mathematica* 28.2 (2022), p. 42. arXiv: `1912.02063 [math.GT]`.

[2] J. Fuchs, I. Runkel, and C. Schweigert. "TFT construction of RCFT correlators I: partition functions". In: *Nuclear Physics B* 646.3 (2002), pp. 353–497. arXiv: `hep-th/0204148`.

[3] A. Hofer and I. Runkel. "Modular functors from non-semisimple 3d TFTs". in preperation.