

# Non-semisimple CFT/TFT correspondence II: Orbifold theories

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This article is a continuation of [HR2] and contains explicit computations for how the construction of a full 2d CFTs from surface defects in a non-semisimple 3d TFTs works for the class of defects obtained via the generalised orbifold procedure of [CRS1; CRS2].

In this setting I show that the field objects, together with the algebra structures on them induced by genus zero correlators, recovers the structure proposed in [FS3]. Moreover, I generalise the notion of Cardy algebras, introduced in [KR], to the non-semisimple setting and show that bulk and boundary field objects provide an example thereof.

Finally, I study the algebra of operators generated by topological defects acting on the object of bulk fields and show that it is isomorphic to the linearised Grothendieck ring of the category of topological line defects of the full CFT, providing a vast source of examples for non-invertible and non-semisimple topological symmetries.

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# 1 Introduction

## Acknowledgements

I want to thank Nils Carqueville, Tyler Franke, Ingo Runkel for useful discussions as well as helpful comments on an earlier draft. This work was initiated at the University of Hamburg during my doctoral research where it was supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy - EXC 2121 "Quantum Universe" - 390833306, and by the Collaborative Research Centre CRC 1624 "Higher structures, moduli spaces and integrability" - 506632645. Most of it was carried out at the Max Planck Institute for Mathematics in Bonn, whose support most is appreciated.

## Conventions

Throughout this article,  $\mathcal{C}$  will be a fixed modular tensor category. The relevant Moreover, we will appeal to the standard coherence results and assume  $\mathcal{C}$  to be strictly monoidal and strictly pivotal. We will always work over an algebraically closed field  $\mathbb{k}$  of characteristic zero. Unless otherwise noted, functors between abelian (and linear) categories will always be assumed to be additive (and linear). For details see Section 2.

With manifold we will always mean a compact smooth manifold, however since we are exclusively working in dimensions less than four we will sometimes work in the topological category instead. Every manifold we will consider will be oriented, and every diffeomorphism will be orientation preserving unless explicitly stated otherwise. For any manifold  $M$  we will denote the manifold with reversed orientation by  $-M$ . The interval  $[0, 1]$  will be denoted by  $I$  and the unit circle by  $S^1$ . Finally, by a closed manifold we mean a compact manifold without boundary.

In this article a 2-category will always mean a weak 2-category otherwise known as a bicategory. Analogously a 2-functor will always mean a weak 2-functor with coherence isomorphisms otherwise known as a pseudofunctor.

## 2 Background

In this section, we collect definitions and results related to ribbon categories with a particular emphasis on modular tensor categories  $\mathcal{C}$ . Moreover, we collect a number of results on relations between  $\mathcal{C}$ , its Drinfeld centre  $\mathcal{Z}(\mathcal{C})$ , and the Deligne product  $\mathcal{C} \boxtimes \bar{\mathcal{C}}$  in the modular setting.

## 2.1 Algebraic background

A linear category is called *finite* if it is equivalent, as a linear category, to the category  $A\text{-mod}$  of finite dimensional modules of some finite dimensional algebra  $A$ . In particular, a finite linear category is abelian, every object has a projective cover, and its Hom sets are finite dimensional vector spaces. For the complete intrinsic definition of finite linear categories see [EGNO, Sec. 1.8]. By a *finite tensor category* we mean a finite linear category which is in addition a rigid monoidal category such that the monoidal product  $\otimes$  is bilinear and the monoidal unit  $\mathbb{1}$  is simple.

A *finite ribbon category*  $\mathcal{C}$  is a finite tensor category which is also ribbon. We will employ the following conventions for structure morphisms in  $\mathcal{C}$ . We will use the same conventions as [TV] and denote with  $\overline{\mathcal{C}}$  the same underlying pivotal tensor category, but equipped with the inverse braiding and twist.

We will denote the Lyubashenko coend of  $\mathcal{C}$  with

$$L := \int^{X \in \mathcal{C}} X^* \otimes X \quad (2.1)$$

with universal dinatural transformation

$$\iota_X: X^* \otimes X \rightarrow L. \quad (2.2)$$

The object  $L$  is referred to as the *canonical coend* and naturally carries the structure of a Hopf algebra with Hopf pairing in the braided monoidal category  $\mathcal{C}$ . For Hopf algebras in braided monoidal categories we will follow the convention of [TV, Ch. 6]. The structural morphisms of  $L$  are induced from the universal property of the coend as shown in Figure 1. If we do not assume  $\mathcal{C}$  to be strictly pivotal the canonical isomorphisms  $(X \otimes Y)^* \cong Y^* \otimes X^*$  for the product,  $\mathbb{1}^* \otimes \mathbb{1} \cong \mathbb{1}$  for the unit, and  $X \cong X^{**}$  for the antipode are needed, see [TV, Sec. 6.4 & 6.5] for details. Exchanging the over and under braidings in the definition of  $\omega$  gives a pairing  $\overline{\omega}: L \otimes L \rightarrow \mathbb{1}$  which satisfies

$$\omega \circ (S \otimes \text{id}_L) = \overline{\omega} = \omega \circ (\text{id}_L \otimes S) \quad (2.3)$$

see [KL, Eq. 5.2.8].

**Definition 2.1.** A finite ribbon category  $\mathcal{C}$  is called *modular* if the canonical Hopf pairing  $\omega$  of the coend  $L$  is non-degenerate.

There are other equivalent definitions of modularity, one of them will be discussed below, see [Shi2, Thm. 1.1] for the full list. From now on we will always assume  $\mathcal{C}$  to be modular. It can be shown that  $\mathcal{C}$  is unimodular and that the Hopf algebra  $L$

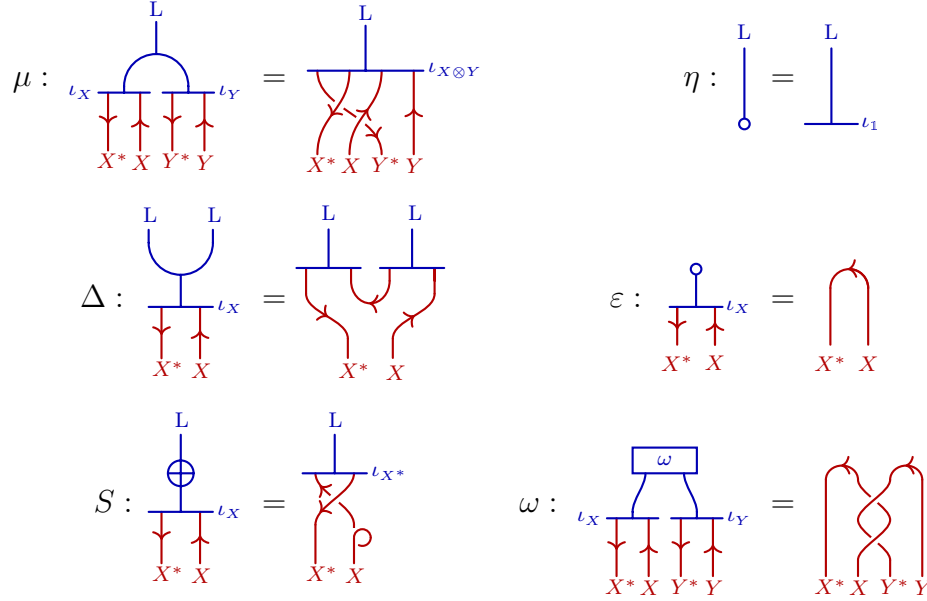


Figure 1: The dinatural transformations defining the Hopf algebra structure morphisms and the Hopf pairing on  $L$ . We use red for the ribbons labelled with  $X; Y \in \mathcal{C}$  here to highlight that the structural morphism of  $L$  do not depend on the specific labels.

admits a unique-up-to-scalar two-sided integral  $\Lambda: \mathbb{1} \rightarrow L$  and cointegral  $\lambda: L \rightarrow \mathbb{1}$ , see [DGGPR, Sec. 2] and references therein for more details.

We will normalise the integral and cointegral in terms of the *modularity parameter*  $\zeta \in \mathbb{k}^\times$  as

$$\lambda \circ \Lambda = \text{id}_{\mathbb{1}} \quad , \quad \omega \circ (\text{id}_L \otimes \Lambda) = \zeta \lambda \quad . \quad (2.4)$$

That the second condition is possible is in fact equivalent to non-degeneracy of  $\omega$ , see [Ker, Thm. 5] or [TV, Lem. 6.2]. The following statement is analogous to the case of classical Hopf algebras [KL, Cor. 4.2.13].

**Proposition 2.2.** The cointegral  $\lambda$  induces a non-degenerate pairing  $\kappa := \lambda \circ \mu: L \otimes L \rightarrow \mathbb{1}$  which equips  $L$  with the structure of a Frobenius algebra in  $\mathcal{C}$ . The copairing is given by  $(S \otimes \text{id}_L) \circ \Delta \circ \Lambda: \mathbb{1} \rightarrow L \otimes L$ .

The pairing  $\kappa$  is called the *Radford pairing*. Note that this convention for  $\kappa$  keeps the algebra structure of  $L$  fixed while the one we used in [HR1] keeps the coalgebra

structure. Since the Hopf pairing is also non-degenerate, the composition

$$\mathcal{S} := (\omega \otimes \text{id}_L) \circ (\text{id}_L \otimes ((S \otimes \text{id}_L) \circ \Delta \circ \Lambda)) : L \rightarrow L \quad (2.5)$$

is invertible and satisfies

$$\kappa \circ (\mathcal{S} \otimes \text{id}_L) = \omega = \kappa \circ (\text{id}_L \otimes \mathcal{S}) \quad (2.6)$$

by definition. We can define another endomorphism of  $L$  by the universal property via  $\mathcal{T} \circ \iota_X = \iota_X \circ (\text{id}_{X^*} \otimes \vartheta_X)$ . This morphism is invertible with inverse  $\mathcal{T}^{-1} \circ \iota_X = \iota_X \circ (\text{id}_{X^*} \otimes \vartheta_X^{-1})$ . Moreover the constants  $\Delta^\pm$  defined via

$$\varepsilon \circ \mathcal{T}^{\pm 1} \circ \Lambda = \Delta^\pm \text{id}_1 \quad (2.7)$$

are non-zero and satisfy  $\zeta = \Delta^+ \Delta^-$ , see [DGGPR, Prop. 2.6 & Cor. 4.6]. Using this and the normalisation of  $\Lambda$  and  $\lambda$  a direct computation shows that

$$\mathcal{S}^2 = \zeta S^{-1} \quad (2.8)$$

where  $S^{-1}$  is the inverse of the antipode of  $L$ . The name *modular* tensor category is justified by the following lemma:

**Lemma 2.3** ([Lyu1, Thm. 2.1.9]). The morphisms  $\mathcal{T}$  and  $\mathcal{S}$  induce a projective  $\text{SL}(2, \mathbb{Z})$ -action on the morphism spaces  $\text{Hom}_{\mathcal{C}}(L, \mathbb{1})$  and  $\text{Hom}_{\mathcal{C}}(\mathbb{1}, L)$  via pre- and postcomposition, respectively.

Accordingly, we will call  $\mathcal{S} : L \rightarrow L$  the modular  $S$ -transformation of  $\mathcal{C}$ . The modular group action on  $\text{Hom}_{\mathcal{C}}(L, \mathbb{1})$  is actually a part of the chiral modular functor constructed in [HR1].

## 2.2 (Co)modules of the canonical Hopf algebra

Let us now discuss how the categories of  $L$ -modules and comodules are related to more well-known categories.

To this end first recall the *Drinfeld centre* of a monoidal category  $\mathcal{A}$  as the braided monoidal category  $\mathcal{Z}(\mathcal{A})$  with objects pairs  $(X, \gamma)$ , where  $X \in \mathcal{A}$ , and  $\gamma : X \otimes - \Rightarrow - \otimes X$  is a natural isomorphism, called a *half braiding*, satisfying a hexagon type axiom [EGNO, Def. 7.13.1]. A key property of the Drinfeld centre is that for suitable  $\mathcal{A}$  it is modular:

**Proposition 2.4** ([Shi6, Thm. 5.11]). Let  $\mathcal{A}$  be finite tensor category satisfying the sphericity condition of [DSS, Def. 3.5.2]. Then  $\mathcal{Z}(\mathcal{A})$  is a modular tensor category.

In particular a modular tensor category  $\mathcal{C}$  satisfies this sphericity condition which implies that  $\mathcal{Z}(\mathcal{C})$  is a modular tensor category as well. To see this, combine [SS, Thm. 1.3] and [GR, Cor. 4.7]. A generalisation of this result to pivotal tensor categories can be found in [MW, Cor. 2.13].

The forgetful functor  $U: \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$  which sends a pair  $(X, \gamma)$  to  $X$  has a left adjoint  $F: \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C})$ . In the modular setting  $F$  is actually also a right adjoint as we will see below. The corresponding monad  $U \circ F$  is naturally isomorphic to the central monad  $Z$  on  $\mathcal{C}$

$$(U \circ F)(-) \cong Z(-) := \int^{X \in \mathcal{C}} X^* \otimes - \otimes X \quad (2.9)$$

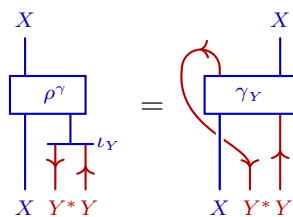
and the corresponding Eilenberg-Moore category is precisely  $\mathcal{Z}(\mathcal{C})$  [BV2, Sec. 5.6] (see also [TV, Ch. 9] for a textbook account). See also [HR1, Prop. 5.9] for a topological proof of this statement using modular functors.

From the braiding in  $\mathcal{C}$  we get an isomorphism of monads  $Z(-) \cong - \otimes L$ , see e.g. [Shi2, Lem. 3.7], which gives rise to the following:

**Lemma 2.5.** The functor

$$\begin{aligned} \mathcal{Z}(\mathcal{C}) &\rightarrow \mathcal{C}_L \\ (X, \gamma) &\mapsto (X, \rho^\gamma) \end{aligned} \quad (2.10)$$

where the right L-action  $\rho^\gamma$  is obtained from the half-braiding  $\gamma$  via



$$(2.11)$$

is an equivalence categories.

This lemma is a special case of [Maj, Thm. 3.2].

Under the equivalence  $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{C}_L$  the forgetful functor  $U: \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$  simply corresponds to the functor forgetting the L-module structure. In particular, its left adjoint is given by the free module functor  $- \otimes L: \mathcal{C} \rightarrow \mathcal{C}_L$ . Moreover, since  $L$  is a Frobenius algebra this is automatically also a right adjoint for  $U$ , as claimed above.

There is also a different functor  $\mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C})$  given by sending an object  $X \in \mathcal{C}$  to the object  $(X, \beta_{X,-}) \in \mathcal{Z}(\mathcal{C})$  with half braiding given by the braiding  $\beta_{X,-}$ . In particular, we can equip every object in  $\mathcal{C}$  with a L-action by combining this functor with the

above equivalence. We will call the resulting action the *canonical L-action on X* and denote it with  $\rho_X: X \otimes L \rightarrow X$ . By taking the (left) partial trace  $\text{tr}_L(\rho_X): L \rightarrow \mathbb{1}$  we get an element in  $\chi_X \in \text{Hom}_{\mathcal{C}}(L, \mathbb{1})$  which is called the *internal character of X* [Shi1, Sec. 3.5].<sup>1</sup>

Next, recall the Deligne tensor product of two finite linear categories  $\mathcal{A}$  and  $\mathcal{B}$  is the finite linear category  $\mathcal{A} \boxtimes \mathcal{B}$  together with a functor  $\boxtimes: \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{A} \boxtimes \mathcal{B}$  which is left exact in each argument and satisfies the following universal property: Let  $\mathcal{D}$  be another finite linear category and let  $F: \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{D}$  be a functor left exact in each argument, then there is a unique (up to unique natural isomorphism) left exact functor  $\hat{F}: \mathcal{A} \boxtimes \mathcal{B} \rightarrow \mathcal{D}$  together with an equivalence  $F \cong \hat{F} \circ \boxtimes$ . See [EGNO, Sec. 1.11] for the existence of the Deligne product as well as more properties.<sup>2</sup> For  $\mathcal{C}$  and  $\mathcal{D}$  modular tensor categories, also  $\mathcal{C} \boxtimes \mathcal{D}$  is a modular tensor category. This can for example be seen by noting that the isomorphism  $L_{\mathcal{C} \boxtimes \mathcal{D}} \cong L_{\mathcal{C}} \boxtimes L_{\mathcal{D}}$  [FSS, Cor. 3.12] where  $L_{\mathcal{C}}$  and  $L_{\mathcal{D}}$  are the canonical Hopf algebra of  $\mathcal{C}$  and  $\mathcal{D}$ , respectively, is an isomorphism of Hopf algebras.

In particular, taking  $\mathcal{D} = \overline{\mathcal{C}}$  shows that  $\mathcal{C} \boxtimes \overline{\mathcal{C}}$  is itself a modular tensor category and we will denote by

$$\mathbb{L} := \int^{Y \in \mathcal{C} \boxtimes \overline{\mathcal{C}}} Y^* \otimes Y \in \mathcal{C} \boxtimes \overline{\mathcal{C}} \quad (2.12)$$

its canonical coend.

Let us now turn to the question of comodules of  $L$ . First note that we can equip any object  $X \in \mathcal{C}$  with the structure of a right  $L$ -comodule with coaction  $\delta_X$  given by

$$\delta_X := (\text{id}_X \otimes \iota_X) \circ (\text{coev}_X \otimes \text{id}_X): X \rightarrow X \otimes L. \quad (2.13)$$

We will call this the *canonical L-coaction on X* and the (left) partial trace  $\check{\chi}_X := \text{tr}_L(\delta_X) = \iota_X \circ \widetilde{\text{coev}}_X: \mathbb{1} \rightarrow L$  the *internal cocharacter of X*.

**Lemma 2.6.** The functor

$$\begin{aligned} \mathcal{C} \boxtimes \overline{\mathcal{C}} &\rightarrow \mathcal{C}^L \\ X \boxtimes Y &\mapsto (X \otimes Y, \text{id}_X \otimes \delta_Y) \end{aligned} \quad (2.14)$$

is an equivalence of categories.

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<sup>1</sup>Shimizu defines internal characters using the canonical end  $\mathcal{E} = \int_{X \in \mathcal{C}} X \otimes X^*$  which can be identified with the dual of the coend  $L$  via the Hopf pairing  $\omega$  and under this identification the two versions of internal characters coincide.

<sup>2</sup>Usually one defines the Deligne tensor product for right exact functors, however by the equivalence of left exact and right exact functors for finite linear categories [FSS, Thm. 3.2] this does not make a difference, see also the discussion in [BW, Sec. 2.4].

This is a special case of [Lyu2, Cor. 2.7.2], see also [EGNO, Prop. 7.18.4] or [Shi4, Rem. 3.6] for a proof based on the theory of exact module categories.

Using the Hopf pairing  $\omega: L \otimes L \rightarrow \mathbb{1}$  we can turn comodules into modules yielding a functor  $\mathcal{C}^L \rightarrow \mathcal{C}_L$ . Under the equivalences of Theorem 2.5 and 2.6 this is precisely the ribbon functor  $\mathcal{C} \boxtimes \bar{\mathcal{C}} \rightarrow \mathcal{Z}(\mathcal{C})$  sending  $X \boxtimes Y \in \mathcal{C} \boxtimes \bar{\mathcal{C}}$  to  $(X \otimes Y, ((\beta_{X,-} \otimes \text{id}_Y) \circ (\text{id}_X \otimes \bar{\beta}_{Y,-})))$ . Moreover, since  $\mathcal{C}$  is assumed to be modular, i.e.  $\omega$  is non-degenerate this functor is an equivalence. Shimizu proved that the converse also holds:

**Proposition 2.7** ([Shi2, Thm. 1.1]). Let  $\mathcal{B}$  be a finite ribbon tensor category. Then  $\mathcal{B}$  is modular if and only if the canonical ribbon functor

$$\mathcal{B} \boxtimes \bar{\mathcal{B}} \rightarrow \mathcal{Z}(\mathcal{B}) \quad (2.15)$$

is an equivalence.

Note that the functor  $\mathcal{C} \boxtimes \bar{\mathcal{C}} \rightarrow \mathcal{C}$  induced by the monoidal product  $\otimes$  on  $\mathcal{C}$  corresponds to the forgetful functor  $\mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$  and thus also has a two-sided adjoint  $\mathcal{C} \rightarrow \mathcal{C} \boxtimes \bar{\mathcal{C}}$  which in this form is given by  $F(X) = \int^{Y \in \mathcal{C}} (Y^* \otimes X) \boxtimes Y \in \mathcal{C} \boxtimes \bar{\mathcal{C}}$ .

For an object  $X \in \mathcal{C}$  the canonical action  $\rho_X$  and coaction  $\delta_X$  satisfy  $\rho_X = (\text{id}_X \otimes \omega) \circ (\delta_X \otimes \text{id}_L)$  which immediately gives  $\chi_X = \omega \circ (\check{\chi}_X \otimes \text{id}_L)$ .

Finally, using yet another characterisation of modularity<sup>3</sup> one can show that the endomorphisms obtained by combining the canonical coaction  $\delta$  with the cointegral  $\lambda$  has the following cutting property:

**Lemma 2.8** ([GR, Lem. 6.3]). For each  $X \in \mathcal{C}$  there exist  $m \geq 0$ ,  $a: X \rightarrow \mathbb{1}^{\oplus m}$ ,  $b: \mathbb{1}^{\oplus m} \rightarrow X$  such that

$$\begin{aligned} (\text{id}_X \otimes \lambda) \circ \delta_X &= b \circ a \\ &= \sum_{\alpha=1}^m b_\alpha \circ a_\alpha \end{aligned} \quad (2.16)$$

where  $a_\alpha: X \rightarrow \mathbb{1}$ ,  $b_\alpha: \mathbb{1} \rightarrow X$  are the components of  $a$ ,  $b$ .

In particular, for non-semisimple  $\mathcal{C}$ , the case  $m = 0$  appears for example for  $X = \mathbb{1}$  since  $(\text{id}_{\mathbb{1}} \otimes \lambda) \circ \delta_{\mathbb{1}} = \lambda \circ \eta = 0$  by a monadic extension of Maschke's theorem [BV1, Thm. 6.5]. Using (2.4) we get a similar statement for the integral  $\Lambda$  as  $\rho_X \circ (\text{id}_X \otimes \Lambda) = \zeta(\text{id}_X \otimes \lambda) \circ \delta_X$ .

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<sup>3</sup>Namely the triviality of the Müger centre [Shi2, Sec. 4].

## 2.3 Algebras and module categories

Let us start with a quick recollection of (pivotal) module categories. We will be very brief here and refer the reader to [EGNO, Ch. 7] as well as [FS2]. A right  $\mathcal{C}$ -module category consists of a finite linear category  $\mathcal{M}$ , a functor  $- \triangleleft -: \mathcal{M} \times \mathcal{C} \rightarrow \mathcal{M}$ , exact in its first variable, as well as a mixed associator and a mixed unit that obey mixed pentagon and triangle axioms. We will abbreviate all of this data and simply write  $\mathcal{M}$  in the following. For two module categories  $\mathcal{M}$  and  $\mathcal{N}$  a module functor from  $\mathcal{M}$  to  $\mathcal{N}$  consists of a functor  $F: \mathcal{M} \rightarrow \mathcal{N}$  together with a natural transformation  $F(- \triangleleft -) \Rightarrow F(-) \triangleleft -$  satisfying a compatibility condition with the associators and unitors. Finally, one can also consider module natural transformations which need to be compatible with the above natural transformation. Altogether this forms a 2-category  $\text{Mod}(\mathcal{C})$ .

There is a close relation between  $\mathcal{C}$ -module categories and algebra objects in  $\mathcal{C}$ . Let  $A \in \mathcal{C}$  be an algebra object. For every left  $A$ -module  $M$  and every  $X \in \mathcal{C}$  we can endow  $M \otimes X$  with the structure of a left  $A$ -module. This endows the category  ${}_A\mathcal{C}$  of left  $A$ -modules in  $\mathcal{C}$  with the structure of a right  $\mathcal{C}$ -module category, see e.g. [EGNO, Sec. 7.8]. Under some technical conditions we also have the converse in the sense that for every (nice enough) module category  $\mathcal{M}$  there exists an algebra  $A \in \mathcal{C}$  such that  $\mathcal{M} \simeq {}_A\mathcal{C}$  as  $\mathcal{C}$ -module categories, see [EGNO, Thm. 7.10.1] for the precise statement. The idea is to obtain the algebra using the *internal* or *inner* Hom functor  $\underline{\text{Hom}}_{\mathcal{M}}(-, -): \mathcal{M}^{\text{op}} \times \mathcal{M} \rightarrow \mathcal{C}$  which is defined via

$$\text{Hom}_{\mathcal{C}}(X, \underline{\text{Hom}}_{\mathcal{M}}(M, N)) \cong \text{Hom}_{\mathcal{M}}(M \triangleleft X, N) \quad (2.17)$$

as the right adjoint of the action functor. Note that the inner Hom exists by the exactness of the action functor in the first argument and is automatically left exact. Using the adjunction we can endow  $\underline{\text{Hom}}_{\mathcal{M}}(M, M)$  with an algebra structure for any  $M \in \mathcal{M}$  [EGNO, Sec. 7.9].

A module category is called *exact* if  $\underline{\text{Hom}}_{\mathcal{M}}(-, -): \mathcal{M}^{\text{op}} \times \mathcal{M} \rightarrow \mathcal{C}$  is exact in each variable.<sup>4</sup> We say an algebra  $A \in \mathcal{C}$  is exact if its corresponding module category  ${}_A\mathcal{C}$  is exact. For exact module categories the above theorem applies without further assumptions. Exactness of a module category, or equivalently the corresponding algebra, can be thought of as a semisimplicity condition relative to the base tensor category [CSZ].

For two exact  $\mathcal{C}$ -module categories  $\mathcal{M}$  and  $\mathcal{N}$  we can endow the category of right exact  $\mathcal{C}$ -module functors  $\mathcal{R}\text{ex}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$  with the structure of an exact  $\mathcal{Z}(\mathcal{C})$ -

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<sup>4</sup>This is equivalent to the definition of exact module category from [EGNO, Sec. 7.5] by combining [Shi7, Def. 3.3] and [Shi7, Lem. 5.3].

module category using the half-braiding, see [FS2, Sec. 4.1] for details. The internal Hom for  $\mathcal{R}\text{ex}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$  as a  $\mathcal{Z}(\mathcal{C})$ -module category is called the object of *internal natural transformations* or *internal Nat* and denoted by  $\underline{\text{Nat}}(F, G) \in \mathcal{Z}(\mathcal{C})$  for  $F, G \in \mathcal{R}\text{ex}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ . For many of the computations in this paper the following result of Fuchs and Schweigert will be instrumental:

**Theorem 2.9.** [FS2, Thm. 9] Let  $\mathcal{C}$  be a finite tensor category and  $\mathcal{M}$  and  $\mathcal{N}$  exact  $\mathcal{C}$ -module categories. For any  $F, G \in \mathcal{R}\text{ex}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$  there is a canonical isomorphism

$$\underline{\text{Nat}}(F, G) \cong \int_{M \in \mathcal{M}} \underline{\text{Hom}}_{\mathcal{N}}(F(M), G(M)). \quad (2.18)$$

in  $\mathcal{Z}(\mathcal{C})$ .

$$\begin{array}{ccc} \text{Hom}_{\mathcal{Z}(\mathcal{C})}((z, \beta), \underline{\text{Nat}}(F, G)) & \xrightarrow{\cong} & \text{Hom}_{\mathcal{R}\text{ex}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})}(F \triangleleft (z, \beta), G) \\ \pi_M \downarrow & & \downarrow p_M \\ \text{Hom}_{\mathcal{C}}(z, \underline{\text{Hom}}_{\mathcal{N}}(F(M), G(M))) & \xrightarrow{\cong} & \text{Hom}_{\mathcal{N}}(F(M) \triangleleft z, G(M)) \end{array}$$

Let us denote with  $\text{Mod}^{\text{ex}}(\mathcal{C})$  the 2-category of exact  $\mathcal{C}$ -module categories, right exact module functors and module natural transformations. The above theorem can be upgraded to an equivalence of 2-categories via an Eilenberg-Watts type argument. More precisely one can show that the 2-functor

$$\begin{aligned} \text{Alg}^{\text{ex}}(\mathcal{C}) &\rightarrow \text{Mod}^{\text{ex}}(\mathcal{C}) \\ A &\mapsto {}_A\mathcal{C} \\ {}_B M_A &\mapsto M \otimes_A - \\ \varphi &\mapsto \varphi \otimes_A - \end{aligned} \quad (2.19)$$

where  $\text{Alg}^{\text{ex}}(\mathcal{C})$  is the 2-category of exact algebras, bimodules and bimodule maps in  $\mathcal{C}$ , is an equivalence, see e.g. [EGNO, Prop. 7.11.1]. In particular we have equivalences  ${}_B\mathcal{C}_A \simeq \mathcal{R}\text{ex}_{\mathcal{C}}({}_A\mathcal{C}, {}_B\mathcal{C})$ , where  ${}_B\mathcal{C}_A$  denotes the category of  $B$ - $A$ -bimodules in  $\mathcal{C}$ , for any pair  $A, B$  of exact algebras in  $\mathcal{C}$ . Using this equivalence  ${}_B\mathcal{C}_A$  inherits a  $\mathcal{Z}(\mathcal{C})$ -action as well. Since  $\mathcal{C}$  is modular this gives us a  $\mathcal{C} \boxtimes \overline{\mathcal{C}}$ -action on  ${}_B\mathcal{C}_A$  which can be shown to be explicitly given by

$$\begin{aligned} {}_B\mathcal{C}_A \times \mathcal{C} \boxtimes \overline{\mathcal{C}} &\rightarrow {}_B\mathcal{C}_A \\ (M, X \boxtimes \overline{X}) &\mapsto X \otimes^+ M \otimes^- \overline{X} \end{aligned} \quad (2.20)$$

where  $X \otimes^+ M \otimes^- \bar{X}$  is  $X \otimes M \otimes \bar{X}$  with  $B$ - $A$ -bimodule structure given by

$$(2.21)$$

We will denote the inner Hom of  ${}_B\mathcal{C}_A$  as a  $\mathcal{C} \boxtimes \bar{\mathcal{C}}$ -module category as  $\underline{\text{Hom}}_{B|A}$ . The object  $Z(A) := \underline{\text{Hom}}_{A|A}(A, A)$  is naturally a commutative algebra and known as the *full centre* of the algebra  $A$  [DKR].

From now on we will be interested in *special symmetric Frobenius algebras* and their module categories. A Frobenius algebra  $(A, \mu, \eta, \Delta, \varepsilon)$  in  $\mathcal{C}$  is called  $\Delta$ -separable if  $\mu \circ \Delta = \text{id}_A$  and it is called symmetric if  $\varepsilon \circ \mu = \varepsilon \circ \mu \circ \beta_{A,A} \circ (\text{id}_A \otimes \vartheta_A)$  where  $\beta$  is the braiding and  $\vartheta$  the ribbon twist of  $\mathcal{C}$ . A *special* symmetric Frobenius algebra is a  $\Delta$ -separable symmetric Frobenius algebra with non-zero categorical dimension or equivalently  $\varepsilon \circ \eta \neq 0$ . On the level of module categories  $\Delta$ -separability ensures exactness [SY, Lem. 5.3] while being symmetric endows it with the structure of a so-called *pivotal* module category. We will not need the precise definition of pivotal module category here and only want to note that this structure makes the inner Hom functor a two-sided adjoint of the action functor, see [Sch, Def. 5.2] and [Shi5, Def. 3.11] for the full definition. In particular, for  $A$  a symmetric Frobenius algebra we can now explicitly describe the inner Hom of  ${}_A\mathcal{C}$  as

$$\underline{\text{Hom}}_A(m, n) \cong m^* \otimes_A n \quad (2.22)$$

where  $- \otimes_A -$  denotes the relative tensor product of  $A$ -modules and we used that  $m^*$  naturally carries the structure of a right  $A$ -module, see e.g. [EGNO, Lem. 7.8.24]. Moreover, it can be shown that for  $A, B$  special symmetric Frobenius algebras  ${}_B\mathcal{C}_A$  inherits the structure of a pivotal module category over  $\mathcal{C} \boxtimes \bar{\mathcal{C}}$  [FS2, Cor. 19].

$$\begin{array}{ccc} \text{Hom}_{A|A}(A, X \otimes^+ A \otimes^- \bar{X}) & \xrightarrow{\cong} & \text{Hom}_L(Z(A), X \otimes \bar{X}) \\ \downarrow & & \downarrow \\ \text{Hom}_A(m, X \otimes^+ m \otimes \bar{X}) & \xrightarrow{\cong} & \text{Hom}_{\mathcal{C}}(\underline{\text{Hom}}_A(m, m), X \otimes \bar{X}) \end{array} \quad (2.23)$$

$$\begin{array}{ccc}
\mathrm{Hom}_{A|A}(X \otimes^+ A \otimes^- \overline{X}, A) & \xrightarrow{\cong} & \mathrm{Hom}_L(X \otimes \overline{X}, Z(A)) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_A(X \otimes^+ m \otimes \overline{X}, m) & \xrightarrow{\cong} & \mathrm{Hom}_C(X \otimes \overline{X}, \underline{\mathrm{Hom}}_A(m, m))
\end{array} \tag{2.24}$$

### 3 Non-semisimple defect TFT

#### 3.1 Surface defects from algebras

Let us now discuss how the algebraic objects discussed above can be used to construct non-trivial surface defects. More concretely, we will now to apply the construction of [CRS1] to the non-semisimple TFT

$$\widehat{V}_C: \widehat{\mathrm{Bord}}_{3,2}^X(\mathcal{C}) \rightarrow \mathrm{vect}_{\mathbb{k}} \tag{3.1}$$

of [DGGPR] to obtain a non-semisimple TFT  $Z_C$  with non-trivial surface defects satisfying the requirements outlined in [HR2, Sec. 5.1].

For our purposes we will only need the case where at most two 2-strata meet at an adjacent 1-stratum. Due to this we will not introduce the full defect data  $\mathbb{D}_C$  of [CRS1, Sec. 5] and instead restrict our attention to the following:

- $D_3 = \{*\}$
- $D_2 = \{\text{special symmetric Frobenius algebras in } \mathcal{C}\}$
- $D_1 = \{\text{Bimodules between special symmetric Frobenius algebras in } \mathcal{C}\}$
- $D_0 = \{\text{Bimodule maps in } \mathcal{C}\}$

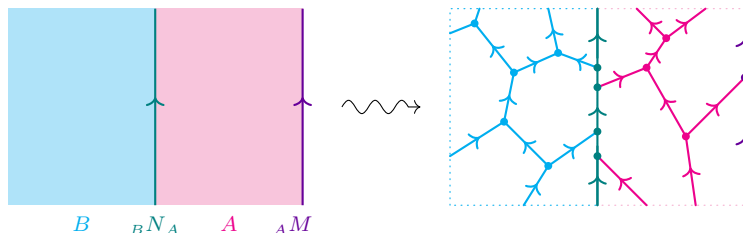
The adjacency maps for bimodules are such that the algebra acting from the right is the source. In terms of higher categories of defects this data is encoded in the 2-category  $\mathrm{FrobAlg}^{\mathrm{sym}, \mathrm{sp}}$  of special symmetric Frobenius algebras, bimodules, and bimodule morphisms. The natural choice for the transparent element  $T \in D_2$  is the monoidal unit  $\mathbb{1}$ . In terms of module categories the transparent element is  $\mathcal{C}$  as a module category over itself.

Next we turn to the construction of the defect TFT

$$Z_C: \mathrm{Bord}_{3,2}^{X, \mathrm{def}}(\mathbb{D}_C) \rightarrow \mathrm{vect}_{\mathbb{k}} \tag{3.2}$$

with this defect data. The idea is to obtain  $Z_{\mathcal{C}}$  by replacing every surface defect with a mesh of line defects labelled by the corresponding algebra object and then evaluating the resulting bordism with line defects with the TFT  $\widehat{V}_{\mathcal{C}}$ . This way of constructing non-trivial surface defects is referred to as a *generalised orbifold* or *internal state sum* or *gauging* construction and goes back to [KS] where it was used to reinterpret the TFT construction of RCFT correlators [FRSI].

Let us make the evaluation process a bit more precise. The mesh of line defects is obtained as follows: First we pick a triangulation (or more generally a cell decomposition) of the surface defect. Next we consider the Poincaré dual graph of the triangulation. We now label edges of the graph with the algebra and the vertices with the multiplication or comultiplication, depending on the orientation. At the vertices where the graph intersects a line defect we use the bi(co)module structure.<sup>5</sup> Schematically this procedure can be visualised as follows:



After evaluating with  $\widehat{V}_{\mathcal{C}}$  the result does not depend on the choice of triangulation because of the  $\Delta$ -separability and the Frobenius relations.

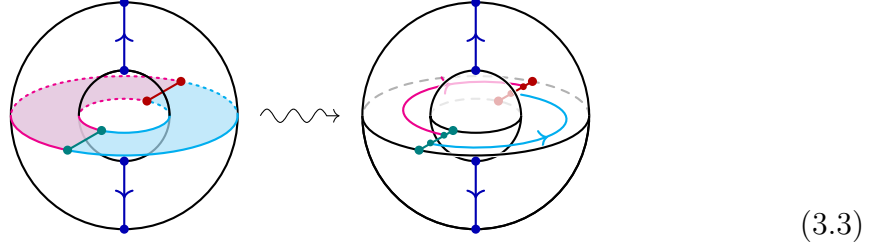
With this we get the action of  $Z_{\mathcal{C}}$  on closed 3-manifolds. For bordisms between non-trivial objects one needs to be more careful because the procedure outlined above is not independent of the induced triangulation on the boundary. To fix this one performs a colimit over all possible boundary triangulations. We will not go into more details on this point or prove that the whole procedure is well-defined and leads to a defect TFT and instead refer the reader to [CRS1, Sec. 5]. Note that for bordisms with only transparent 2-strata we simply get back  $\widehat{V}_{\mathcal{C}}$ , thus the defect TFT  $Z_{\mathcal{C}}(\Sigma)$  is an extension of  $\widehat{V}_{\mathcal{C}}$  in the sense of [HR2, Def. 5.1].

We also want to mention that for any object  $\Sigma \in \text{Bord}_{3,2}^{\chi,\text{def}}(\mathbb{D}_{\mathcal{C}})$  we can compute the state space  $Z_{\mathcal{C}}(\Sigma)$  as the image of the idempotent we get from performing the above procedure on the cylinder  $\Sigma \times I$ . For example for the defect sphere given in

---

<sup>5</sup>Recall here that every (bi)module over a Frobenius algebra is canonically also a (bi)comodule.

[HR2, Fig. 4 (b)] we have:



Note here that the boundary of the bordism on the right is an object in  $\text{Bord}_{3,2}^{\mathcal{X}}(\mathcal{C})$ .

**Remark 3.1.** The defect TFT  $Z_{\mathcal{C}}$  is technically not defined on all of  $\text{Bord}_{3,2}^{\mathcal{X},\text{def}}(\mathbb{D}_{\mathcal{C}})$  as we still need to ensure admissibility in order to evaluate  $\hat{V}_{\mathcal{C}}$ . However, as already mentioned in [HR2, Rem. 5.2], all the bordism we care about here will have an outgoing boundary and will thus be automatically admissible. Due to this we will tacitly ignore this subtlety in the following.

Finally, one can slightly generalise this construction and work with *separable* Frobenius algebras instead of  $\Delta$ -separable ones, see [Mul, Sec. 3.2]. For our purposes  $\Delta$ -separable ones will be sufficient.

## 4 TFT construction of CFT correlators

In this section we will briefly recall, the construction of [HR2, Sec. 5] which uses a suitable 3d defect TFT extending  $\hat{V}_{\mathcal{C}}$  to produce a full 2d CFT with monodromy data governed by the modular functor of  $\mathcal{C}$ . In Section 5 we will then apply this to the orbifold defect TFT  $Z_{\mathcal{C}}$  from Section 3.1 in order to obtain full logarithmic CFTs. The 2-categorical formalism of topological world sheets and modular functors, as well as for all proofs, we refer the reader to [HR2, Sec. 3-5].

### 4.1 Full CFTs as oplax natural transformations

First recall from [HR2, Def. 4.3] that for a chiral modular functor  $\text{Bl}^{\mathcal{X}}: \text{Bord}_{2+\varepsilon,2,1}^{\mathcal{X}} \rightarrow \mathcal{P}\text{rof}_{\mathbb{k}}^{\mathcal{L}^{\text{ex}}}$  and a choice of 2-dimensional defect data  $\mathbb{D}$  with associated  $(2,1)$ -category of topological world sheets  $\mathfrak{WS}(\mathbb{D})$ , a *full CFT* based on the full modular functor  $\text{Bl}: \mathfrak{WS}(\mathbb{D}) \rightarrow \mathcal{P}\text{rof}_{\mathbb{k}}^{\mathcal{L}^{\text{ex}}}$  obtained by pulling back  $\text{Bl}^{\mathcal{X}}$  along the orientation double

functor  $\widehat{(-)}: \mathfrak{WS}(\mathbb{D}) \rightarrow \text{Bord}_{2+\varepsilon,2,1}^X$  is a braided monoidal oplax natural transformation

$$\begin{array}{ccc} & \Delta_{\mathbb{k}} & \\ \mathfrak{WS}(\mathbb{D}_{\mathcal{C}}) & \begin{array}{c} \xrightarrow{\quad} \\ \Downarrow \text{Cor} \\ \xrightarrow{\quad} \end{array} & \mathcal{P}\text{rof}_{\mathbb{k}}^{\mathcal{L}\text{ex}} \\ & \text{Bl}_{\mathcal{C}} & \end{array} \quad (4.1)$$

where  $\Delta_{\mathbb{k}}: \mathfrak{WS}(\mathbb{D}_{\mathcal{C}}) \rightarrow \mathcal{P}\text{rof}_{\mathbb{k}}^{\mathcal{L}\text{ex}}$  denotes the constant 2-functor sending every object to  $\text{vect}_{\mathbb{k}}$ . More concretely this definition encodes the following data: For  $\mathfrak{C} \in \mathfrak{WS}(\mathbb{D}_{\mathcal{C}})$  the 1-morphism component  $\text{Cor}_{\mathfrak{C}}: \text{vect}_{\mathbb{k}} \rightarrow \text{Bl}_{\mathcal{C}}(\mathfrak{C})$  encodes an essentially unique object  $\mathbb{F}_{\mathfrak{C}} \in \text{Bl}_{\mathcal{C}}(\mathfrak{C})$ , the *field content* of  $\mathfrak{C}$ , with  $\text{Cor}_{\mathfrak{C}}(-) \cong \text{Hom}_{\text{Bl}_{\mathcal{C}}(\mathfrak{C})}(\mathbb{F}_{\mathfrak{C}}, -)$ . For a 1-morphism  $\mathfrak{S}: \mathfrak{C} \rightarrow \mathfrak{C}'$  in  $\mathfrak{WS}(\mathbb{D}_{\mathcal{C}})$  the corresponding 2-morphism component is a natural transformation

$$\text{Cor}_{\mathfrak{S}}: \text{Cor}_{\mathfrak{C}'} \otimes_{\mathbb{k}} \text{Cor}_{\mathfrak{C}}^{\dagger} \Rightarrow \text{Bl} \quad (4.2)$$

which is equivalent, via the Yoneda lemma, to an element  $\text{Cor}_{\mathfrak{S}}$  in the vector space of  $\text{Bl}(\mathfrak{S})(\mathbb{F}_{\mathfrak{C}}; \mathbb{F}_{\mathfrak{C}'})$ . The coherence data and axioms of a braided monoidal oplax natural transformation correspond directly to the consistency conditions one expects from a consistent system of correlators.

## 4.2 Full CFT from surface defects

Given a 3d defect TFT  $Z_{\mathcal{C}}$  extending  $\widehat{V}_{\mathcal{C}}$  in the sense of [HR2, Def. 5.1] both the field content and the correlators of a candidate transformation  $\text{Cor}$  are obtained by evaluating  $Z_{\mathcal{C}}$  on the so-called *connecting manifolds*. For  $\mathfrak{C} \in \mathfrak{WS}(\mathbb{D}_{\mathcal{C}})$  with underlying 1-manifold  $\Gamma$  the connecting manifold is given by

$$M_{\Gamma} := \Gamma \times [-1, 1] / (p, t) \sim (p, -t) \text{ for } p \in \partial\Gamma, \quad (4.3)$$

and stratification induced from the embedding  $\Gamma \times \{0\} \subset M_{\Gamma}$ , note that  $\partial M_{\Gamma} \cong \widehat{\Gamma}$ . Now for or  $\underline{X} \in \mathcal{C}^{\times \pi_0(\widehat{\Gamma})}$  we glue in discs with  $\underline{X}$ -labelled, positively oriented marked points along  $\partial M_{\Gamma} \cong \widehat{\Gamma}$  to produce an object  $M_{\Gamma}^{\underline{X}} \in \text{Bord}_{3,2}^{X,\text{def}}(\mathbb{D}_{\mathcal{C}})$ , see Figure 2 for the case of a defect interval and of a defect circle. Evaluating with  $Z_{\mathcal{C}}$  on this can be extended a functor

$$\begin{aligned} \text{cor}_{\Gamma}: \mathcal{C}^{\times \pi_0(\widehat{\Gamma})} &\rightarrow \text{vect}_{\mathbb{k}} \\ \underline{X} &\mapsto Z_{\mathcal{C}}(M_{\Gamma}^{\underline{X}}), \end{aligned} \quad (4.4)$$

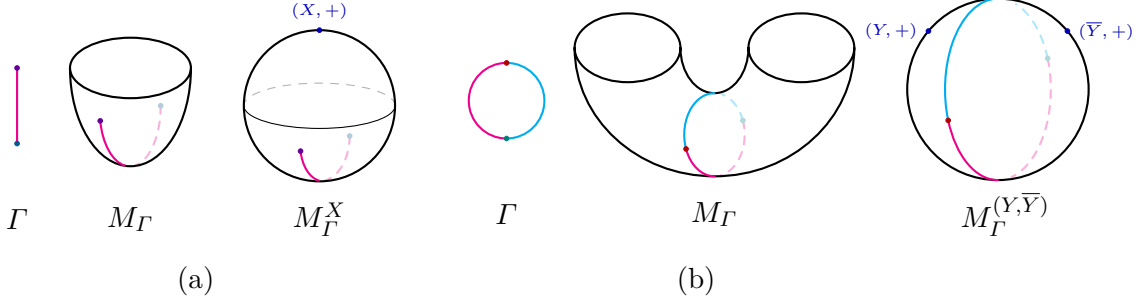


Figure 2: Connecting manifold  $M_\Gamma$  for  $\Gamma$  an interval (a) and a defect circle (b) with and without disc(s) glued in.

together with a counterpart  $\text{cor}_\Gamma^\dagger$  obtained by using  $-M_\Gamma$  with negatively oriented marked points instead. Under Assumption 1 below these functors can be extended to the Deligne product and give the 1-morphism component  $\text{Cor}_\mathfrak{C}$  together with its adjoint profunctor  $\text{Cor}_\mathfrak{C}^\dagger$ .

Similarly, for a world sheet  $\mathfrak{S}: \mathfrak{C} \rightarrow \mathfrak{C}'$  with underlying open-closed bordism  $\Sigma: \Gamma \rightarrow \Gamma'$  one first constructs

$$M_\Sigma := \Sigma \times [-1, 1] / (p, t) \sim (p, -t) \text{ for } p \in \partial^f \Sigma, \quad (4.5)$$

with  $\mathfrak{S}$  as a defect at  $\Sigma \times \{0\}$ . By gluing cylinders over  $M_\Gamma^X$  and  $M_{\Gamma'}^Y$  into parts of the boundary of  $M_\Sigma$  we obtain a bordism

$$M_\Sigma^{X,Y}: M_{\Gamma'}^Y \sqcup M_\Gamma^X \rightarrow \widehat{\Sigma}^{X,Y} \quad (4.6)$$

where  $\widehat{\Sigma}^{X,Y}$  is obtained from the orientation double of  $\mathfrak{S}$  by gluing in disks, see Figure 3 for an illustration. The correlator  $\text{Cor}_\mathfrak{S}$  is now defined as the lift to the Deligne product of the natural transformation with components  $Z_\mathcal{C}(M_\Sigma^{X,Y})$ .

The above construction indeed produces a full CFT under four further assumptions on  $Z_\mathcal{C}$ , see below.

**Theorem 4.1.** [HR2, Thm. 5.9]

Let

$$Z: \text{Bord}_{3,2}^{X,\text{def}}(\mathbb{D}) \rightarrow \text{vect}_k \quad (4.7)$$

be a 3d defect TFT extending the 3d TFT with embedded ribbon graphs

$$\widehat{V}_\mathcal{C}: \widehat{\text{Bord}}_{3,2}^X(\mathcal{C}) \rightarrow \text{vect}_k \quad (4.8)$$

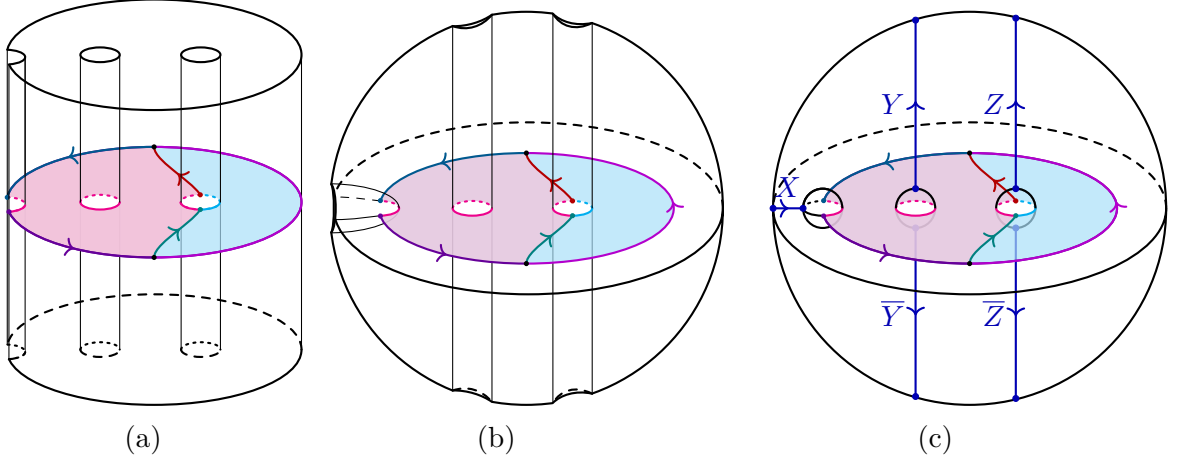


Figure 3: Schematic construction of the connecting bordism  $M_{\Sigma}^{X,Y}$  for  $\mathfrak{S}$  the 1-morphism in  $\mathfrak{WS}(\mathbb{D}_{\mathcal{C}})$  from [HR2, Eq. 3.27]: (a) cylinder over  $\Sigma$  with  $\mathfrak{S}$  as surface defect, (b) the connecting manifold  $M_{\Sigma}$  with corners, (c) connecting bordism  $M_{\Sigma}^{X,Y}$ ;

of [DGGPR] in the sense of [HR2, Def. 5.1]. If  $Z_{\mathcal{C}}$  satisfies Assumptions 1, 2, 3, and 4 then evaluating it on the connecting manifolds defines a full conformal field theory based on  $\text{Bl}_{\mathcal{C}}$ , i.e. a braided monoidal oplax natural transformation

$$\begin{array}{ccc}
 & \Delta_{\mathbb{k}} & \\
 \mathfrak{WS}(\mathbb{D}_{\mathcal{C}}) & \begin{array}{c} \xrightarrow{\quad} \\ \Downarrow \text{Cor} \\ \xrightarrow{\quad} \end{array} & \mathcal{P}\text{rof}_{\mathbb{k}}^{\mathcal{L}\text{ex}} \\
 & \text{Bl}_{\mathcal{C}} & 
 \end{array} . \quad (4.9)$$

Before we continue let us first briefly comment on the additional assumptions on  $Z_{\mathcal{C}}$  and where they arise in the proof of Theorem 4.1.

First note that two of the coherence axioms our candidate  $\text{Cor}$  needs to satisfy require no further input: Naturality of the 2-morphism components of  $\text{Cor}$  under mapping classes, as well as its monoidal structure, follow automatically from functoriality, respectively monoidality, of  $Z_{\mathcal{C}}$ ; see [HR2, Sec. 5.3 & 5.4]. What remains are purely algebraic condition on  $Z_{\mathcal{C}}$ .

First of all, the functors  $\text{cor}_{\Gamma}$  and  $\text{cor}_{\Gamma}^{\dagger}$  are not necessarily left exact and thus only defined on the cartesian product. Thus we need to require:

**Assumption 1.** *For any  $\mathfrak{C} \in \mathfrak{WS}(\mathbb{D}_{\mathcal{C}})$ , the functors  $\text{cor}_{\Gamma}: \mathcal{C}^{\times \pi_0(\hat{\Gamma})} \rightarrow \text{vect}_{\mathbb{k}}$  and  $\text{cor}_{\Gamma}^{\dagger}: (\mathcal{C}^{\text{op}})^{\times \pi_0(\hat{\Gamma})} \rightarrow \text{vect}_{\mathbb{k}}$  are linear and left exact in each variable.*

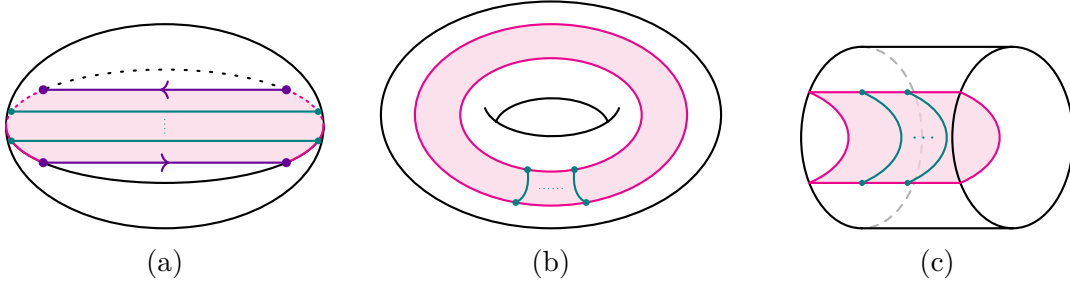


Figure 4: The unit morphism  $\eta_{M_\Gamma} : \emptyset \rightarrow -M_{\Gamma'} \sqcup_{\hat{F}} M_{\Gamma'}$  for  $\mathfrak{C}$  a defect interval (a) and a defect circle (b). As well as a cylinder (c) cut out of (b) to illustrate the stratification.

As the notation suggests we further assume the extensions of  $\text{cor}_\Gamma$  and  $\text{cor}_\Gamma^\dagger$  to the Deligne product to be adjoint profunctors as in [HR2, Lem. 4.1].

Under Assumption 1 the remaining axioms of a braided monoidal oplax natural transformation amount to identifying the structure maps of the adjunction  $\text{Cor}_\mathfrak{C} \dashv \text{Cor}_\mathfrak{C}^\dagger$  with  $Z_\mathfrak{C}$  evaluated on certain bordisms. Oplax naturality, i.e. the factorisation of correlators under gluing of world sheets, requires that both the universal dinatural transformation exhibiting the horizontal composition  $\text{Cor}_\mathfrak{C}^\dagger \diamond \text{Cor}_\mathfrak{C}$  as a coend, and the unit of the adjunction, arise in this way:

**Assumption 2.** For  $\mathfrak{C} \in \mathfrak{WS}(\mathbb{D}_\mathfrak{C})$  a defect interval or circle, the universal dinatural transformation  $\iota_Y : \text{Cor}_\mathfrak{C}(Y)^\dagger \otimes_k \text{Cor}_\mathfrak{C}(Y) \rightarrow \text{Cor}_\mathfrak{C}^\dagger \diamond \text{Cor}_\mathfrak{C}$  is induced by evaluating  $Z_\mathfrak{C}$  on the family of gluing bordisms

$$\tilde{G}_Y : -M_\Gamma^Y \sqcup M_\Gamma^Y \rightarrow -M_\Gamma \sqcup_{\hat{F}} M_\Gamma \quad (4.10)$$

in  $\text{Bord}_{3,2}^{\chi,\text{def}}(\mathbb{D}_\mathfrak{C})$ .

**Assumption 3.** For  $\mathfrak{C} \in \mathfrak{WS}(\mathbb{D}_\mathfrak{C})$  connected the components of the unit  $\eta_{\text{Cor}_\mathfrak{C}} : \text{id}_{\text{vect}_k} \Rightarrow \text{Cor}_\mathfrak{C}^\dagger \diamond \text{Cor}_\mathfrak{C}$  of the adjunction  $\text{Cor}_\mathfrak{C} \dashv \text{Cor}_\mathfrak{C}^\dagger$  is given by the linear map  $Z_\mathfrak{C}(\eta_{M_\Gamma})$  with  $\eta_{M_\Gamma} : \emptyset \rightarrow -M_\Gamma \sqcup_{\hat{F}} M_\Gamma$  as in Figure 4 under the isomorphism  $\text{Hom}_{\text{Bl}_\mathfrak{C}(\mathfrak{C})}(\mathbb{F}_\mathfrak{C}, \mathbb{F}_\mathfrak{C}) \cong Z_\mathfrak{C}(-M_\Gamma \sqcup_{\hat{F}} M_\Gamma)$ .

Finally, oplax unitality, i.e. *non-degeneracy* of 2-point correlators, requires the counit of the adjunction to arise in the same fashion, namely from the connecting bordism of the identity world sheet  $\text{id}_\mathfrak{C}$ :

**Assumption 4.** For any  $\mathfrak{C} \in \mathfrak{WS}(\mathbb{D}_\mathfrak{C})$ , the components of the counit  $\varepsilon_{\text{Cor}_\mathfrak{C}} : \text{Cor}_\mathfrak{C} \otimes_k \text{Cor}_\mathfrak{C}^\dagger \Rightarrow \text{Bl}(\text{id}_\mathfrak{C})$  of the adjunction  $\text{Cor}_\mathfrak{C} \dashv \text{Cor}_\mathfrak{C}^\dagger$  are given by  $Z_\mathfrak{C}(M_{\Gamma \times I}^{X,Y})$ .

With Assumptions 1, 2, 3, and 4 in hand we now turn to Theorem 4.1's intended application.

## 5 Orbifold CFTs

Let us now come to the main results of this article, the explicit computations of various CFT quantities built from orbifold defects. As in [HR2, Sec. 6] we first need to check that  $Z_{\mathcal{C}}$  satisfies the conditions of Theorem 4.1.

**Proposition 5.1.** The orbifold defect TFT

$$Z_{\mathcal{C}}: \text{Bord}_{3,2}^{\chi,\text{def}}(\mathbb{D}_{\mathcal{C}}) \rightarrow \text{vect}_{\mathbb{k}} \quad (5.1)$$

from Section 3.1 satisfies the conditions of Theorem 4.1.

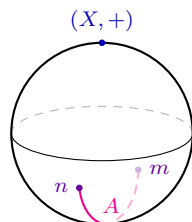
For Assumption 1 we essentially have to compute state spaces of defect spheres. This will be done in the next subsection where we will also verify Assumption 3. The proof of Assumption 2 is completely analogous to [HR1, Prop. 4.8] and will not be discussed further. Finally, Assumption 4 will be treated in detail in Section 5.2.

### 5.1 Field content

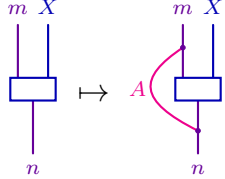
We will be interested in the boundary and disorder fields corresponding to the field content for an interval and a defect circle with two point defects, respectively. Moreover, we can also restrict our attention to the setting of a defect interval with two point defects in its interior as every other case can be obtained by fusing the defects.

#### 5.1.1 Boundary fields

Let  $A \in \text{FrobAlg}^{\text{sym},\text{sp}}$ , and let  $m, n \in {}_A\mathcal{C}$ . Denote with  $I_{n,m}^A \in \mathfrak{WS}_{\mathcal{C}}$  the defect interval with  $A$  labelling the interior  $(0, 1) \subset I$ ,  $m$  labelling  $\{0\} \subset I$ , and  $n$  labelling  $\{1\} \subset I$ . For  $X \in \mathcal{C}$  the connecting manifold  $M_{I_{n,m}^A}^X$  is given by:



As explained in Section 3.1 in order to get the state space  $Z_{\mathcal{C}} \left( M_{I_{n,m}^A}^X \right)$  we have to compute the image of an idempotent on  $\widehat{V}_{\mathcal{C}} \left( M_{I_{n,m}^A}^X \right) \cong \text{Hom}_{\mathcal{C}}(n, X \otimes m)$ . Going through the orbifold procedure this idempotent is given by

$$P: \text{Hom}_{\mathcal{C}}(n, m \otimes X) \rightarrow \text{Hom}_{\mathcal{C}}(n, m \otimes X)$$

(5.2)

where we use the  $A$ -action on  $m$  and the  $A$ -coaction on  $n$ . Using the Frobenius relations and  $\Delta$ -separability it is straightforward to verify that  $\text{im}(P) \cong \text{Hom}_A(n, m \otimes X) \subset \text{Hom}_{\mathcal{C}}(n, m \otimes X)$ . Finally, recall that  $\mathcal{C}_A$  is a right  $\mathcal{C}$ -module category with action given by tensoring with an object from the right. With this we get

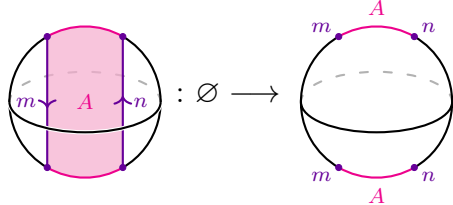
$$\begin{aligned} Z_{\mathcal{C}} \left( M_{I_{n,m}^A}^X \right) &\cong \text{Hom}_A(n, m \otimes X) \\ &\cong \text{Hom}_{\mathcal{C}}(\underline{\text{Hom}}_A(m, n), X) \end{aligned}$$
(5.3)

where we used that the inner hom functor  $\underline{\text{Hom}}_A$  is a two-sided adjoint of the action functor in our setting. We can now read off the boundary field content as

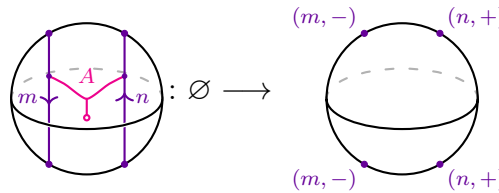
$$\mathbb{F}_{I_{n,m}^A} = \underline{\text{Hom}}_A(m, n) \in \mathcal{C}.$$
(5.4)

Now to verify Assumption 1 for defect intervals we note that  $Z_{\mathcal{C}} \left( M_{I_{n,m}^A}^X \right) \cong \text{Hom}_A(n, m \otimes X)$  is clearly linear and left exact in  $X$ .

For Assumption 3 we have to compute the image of

$$\eta_{M_{I_{n,m}^A}^A} = \left( \text{Diagram} \right) : \emptyset \rightarrow \left( \text{Diagram} \right)$$

(5.5)

under the TFT  $Z_{\mathcal{C}}$ . Going through the orbifold procedure we arrive at the bordism

$$\left( \text{Diagram} \right) : \emptyset \rightarrow \left( \text{Diagram} \right)$$

(5.6)

in  $\widehat{\text{Bord}}_{3,2}^X(\mathcal{C})$ . Applying  $\widehat{V}_{\mathcal{C}}$  to this leads us to the following morphism

$$P_{n,m^*}^A := \text{diagram} : m^* \otimes n \rightarrow m^* \otimes n \quad (5.7)$$

in  $\mathcal{C}$ . Now by using  $\Delta$ -separability of  $A$  it is straightforward to verify that  $P_{n,m^*}^A$  is an idempotent in  $\mathcal{C}$ . Moreover, from [CR, Lem. 2.3] we know that the image of this idempotent is the relative tensor product

$$\mathrm{im}(P_{n,m^*}^A) \cong m^* \otimes_A n \quad (5.8)$$

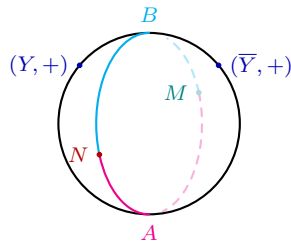
of  $A$ -modules. Finally, using the relation between the inner Hom and the relative tensor product

$$\underline{\mathrm{Hom}}_A(m, n) \cong m^* \otimes_A n \quad (5.9)$$

we arrive at  $Z_{\mathcal{C}}(\eta_{M_{I_{n,m}^A}}) = \text{id}_{\underline{\text{Hom}}_A(m,n)}$ .

### 5.1.2 Bulk and disorder fields

Next we want to study the bulk and disorder fields. To this end let  $A, B \in \text{FrobAlg}^{\text{sym}, \text{sp}}$  and let  $M, N \in {}_B\mathcal{C}_A$ . We denote with  $S_{N,M}^1$  the defect circle with two 1-strata labelled by  $A$  and  $B$ , respectively, one positively oriented 0-stratum labelled with  $M$ , and one negatively oriented 0-stratum labelled with  $N$ . For  $(X, \overline{X}) \in \mathcal{C} \times \overline{\mathcal{C}}$  the corresponding connecting manifold  $M_{S_{N,M}^1}^{(X, \overline{X})}$  is given by the following defect sphere:



In analogy to before we now need to understand the linear idempotent:

$$Q_{N,M}: \text{Hom}_{\mathcal{C}}(N, X \otimes M \otimes \bar{X}) \rightarrow \text{Hom}_{\mathcal{C}}(N, X \otimes M \otimes \bar{X})$$
(5.10)

The image of  $Q_{N,M}$  is isomorphic to  $\text{Hom}_{B|A}(N, X \otimes^+ M \otimes^- \bar{X})$  where  $X \otimes^+ M \otimes^- \bar{X}$  is the  $B$ - $A$ -bimodule with underlying object  $X \otimes M \otimes \bar{X}$  and bimodule structure as in (2.21). To compute the field content we use that  ${}_B\mathcal{C}_A$  is a pivotal  $\mathcal{C} \boxtimes \bar{\mathcal{C}}$ -module category giving:

$$\begin{aligned} Z_{\mathcal{C}} \left( M_{S_{N,M}^1}^{(X, \bar{X})} \right) &\cong \text{Hom}_{B|A}(N, X \otimes^+ M \otimes^- \bar{X}) \\ &= \text{Hom}_{B|A}(N, M \triangleleft (X \boxtimes \bar{X})) \\ &\cong \text{Hom}_{\mathcal{C} \boxtimes \bar{\mathcal{C}}}(\underline{\text{Hom}}_{B|A}(M, N), X \boxtimes \bar{X}). \end{aligned} \tag{5.11}$$

Thus, we have

$$\mathbb{F}_{S_{N,M}^1} = \underline{\text{Hom}}_{B|A}(M, N) \in \mathcal{C} \boxtimes \bar{\mathcal{C}}. \tag{5.12}$$

Moreover, using the Eilenberg-Watts equivalence  ${}_B\mathcal{C}_A \simeq \text{Rex}_{\mathcal{C}}({}_A\mathcal{C}, {}_B\mathcal{C})$  we get

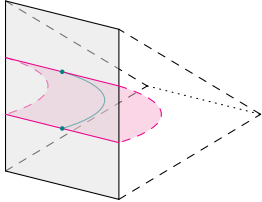
$$\underline{\text{Hom}}_{B|A}(M, N) \cong \underline{\text{Nat}}(M \otimes_A -, N \otimes_A -) \tag{5.13}$$

and thus obtain the proposed field content of [FS3]. In Section 5.3 we will show that certain genus zero correlators naturally endow these objects also with the proposed algebraic structures.

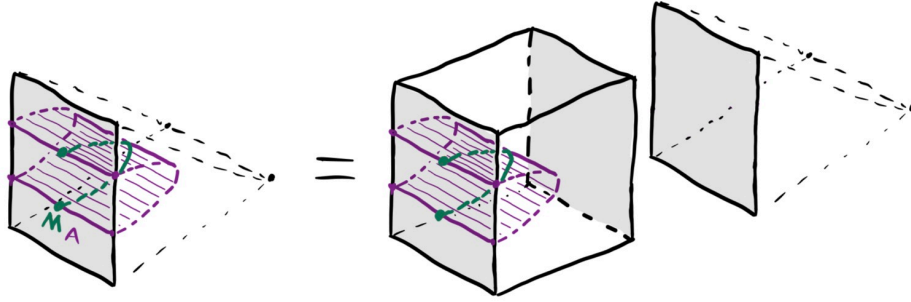
Finally, note that  $\text{Hom}_{\mathcal{C} \boxtimes \bar{\mathcal{C}}}(\underline{\text{Hom}}_{B|A}(M, N), X \boxtimes \bar{X})$  is clearly linear and left exact in  $X$  and  $\bar{X}$  thus verifying Assumption 1 also for defect circles. Using monoidality of  $Z_{\mathcal{C}}$  this completely verifies Assumption 1.

Before we continue note that  $\underline{\text{Hom}}_{B|A}(M, N) \cong \underline{\text{Hom}}_{A|A}(A, M^* \otimes_B N)$ . This can be seen as the result of fusing the  $M$  and  $N$  defect. Thus, from now on we will restrict our attention to the defect circle  $S_{A,M}^1$  with one  $A$ -labelled 1-stratum and one negatively (TODO: check if this makes sense) oriented 0-stratum with label  $M \in {}_A\mathcal{C}_A$ .

For Assumption 3 we have to compute the image of

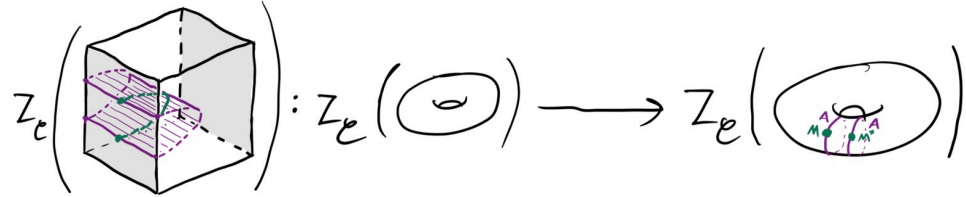
$$\eta_{M_{S^1_{A,M}}} \quad (5.14)$$


under  $Z_C$ . Instead of computing the image of  $\eta_{M_{S^1_{A,M}}}$  directly it is useful to first decompose it as



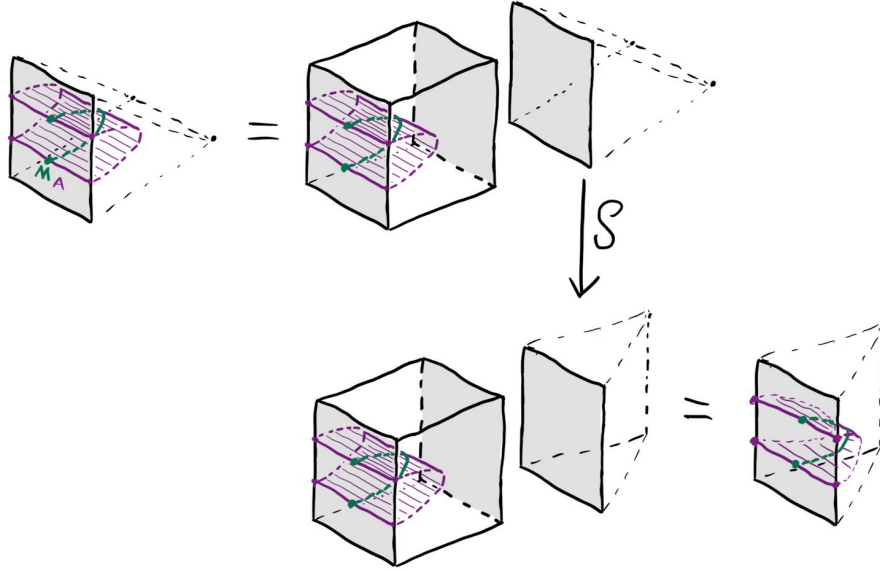
Note that this decomposition is slightly ambiguous if we don't specify the boundary parametrisations used to

From this we see that we can instead equivalently study the linear map

$$Z_C \left( \text{cube with pink surface} \right) : Z_C \left( \text{torus} \right) \longrightarrow Z_C \left( \text{torus with pink surface} \right)$$


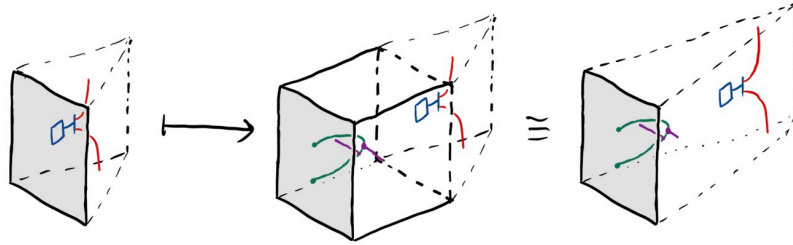
**TODO:** add comments about different solid tori

**Remark 5.2.** The apparent ambiguity in the target of bordism can be understood the first Instead of gluing with the identity we can glue with using the modular  $\mathcal{S}$ -transformation of the torus. This results in a changing which of the generators of  $\pi_1(T^2)$  is contractible in the solid torus



Going through the orbifold procedure we arrive at the following linear map:

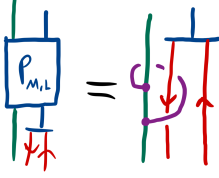
$$\hat{v}_e(\bigcirc) \longrightarrow \hat{v}_e\left(\bigcirc \begin{matrix} \bullet \\ (m_1, \dots) \end{matrix} \begin{matrix} \bullet \\ (m_1, \dots) \end{matrix}\right)$$



Which under the isomorphism of TFT state spaces with hom-spaces in  $\mathcal{C}$  becomes

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(\mathbb{L}, \mathbb{1}) &\rightarrow \text{Hom}_{\mathcal{C}}(M \otimes \mathbb{L}, M) \\ f &\mapsto (\text{id}_M \otimes f) \circ P_{M, \mathbb{L}} \end{aligned} \tag{5.15}$$

where  $P_{M, \mathbb{L}}: M \otimes \mathbb{L} \rightarrow M \otimes \mathbb{L}$  is the unique morphism defined by



**Lemma 5.3.** The morphism  $P_{M,L}: M \otimes L \rightarrow M \otimes L$  is an idempotent in  $\mathcal{C}^L$ . Moreover, its image  $\text{im}(P_{M,L})$  is isomorphic to  $\underline{\text{Hom}}_{A|A}(A, M)$ .

*Proof.* First note that  $P_{M,L}$  commutes with the comultiplication of  $L$  and thus is a morphism in  $\mathcal{C}^L$ . Idempotency follows from a direct computation, see also [FRSI, Lem. 5.2].

For the second part, we will show that  $\text{im}(P_{M,L})$  satisfies the universal property of  $\underline{\text{Hom}}_{A|A}(A, M)$ , i.e. that there is a natural isomorphism

$$\text{Hom}_{A|A}(A \triangleleft X \boxtimes \overline{X}, M) \cong \text{Hom}_L(X \otimes \overline{X}, \text{im}(P_{M,L})). \quad (5.16)$$

First note that by definition we have

$$\text{Hom}_{A|A}(A \triangleleft X \boxtimes \overline{X}, M) \subset \text{Hom}_A(A \otimes X \otimes \overline{X}, M) \quad (5.17)$$

as well as

$$\text{Hom}_L(X \otimes \overline{X}, \text{im}(P_{M,L})) \subset \text{Hom}_L(X \otimes \overline{X}, M \otimes L). \quad (5.18)$$

Moreover, by [FFRS, Lem. 2.4] we can identify  $\text{Hom}_L(X \otimes \overline{X}, \text{im}(P_{M,L}))$  with the subspace

$$\{f \in \text{Hom}_L(X \otimes \overline{X}, M \otimes L) \mid P_{M,L} \circ f = f\}. \quad (5.19)$$

Next recall the following two free-forgetful adjunctions:

1. For  $A$ -modules

$$\begin{aligned} \text{Hom}_C(X \otimes \overline{X}, M) &\cong \text{Hom}_A(X \otimes \overline{X} \otimes A, M) \\ f &\mapsto \lambda_M \circ (f \otimes \text{id}_A) \\ g \circ (u \otimes \text{id}_{X \otimes \overline{X}}) &\leftarrow g \end{aligned} \quad (5.20)$$

where  $\lambda_M: A \otimes M \rightarrow M$  is the left  $A$ -action on  $M$  and  $u: \mathbb{1} \rightarrow A$  is the unit of  $A$ .

2. For  $L$ -comodules

$$\begin{aligned} \text{Hom}_L(X \otimes \overline{X}, M \otimes L) &\cong \text{Hom}_C(X \otimes \overline{X}, M) \\ f &\mapsto (\text{id}_M \otimes \varepsilon) \circ f \\ (g \otimes \text{id}_L) \circ (\text{id}_X \otimes \delta_{\overline{X}}) &\leftarrow g \end{aligned} \quad (5.21)$$

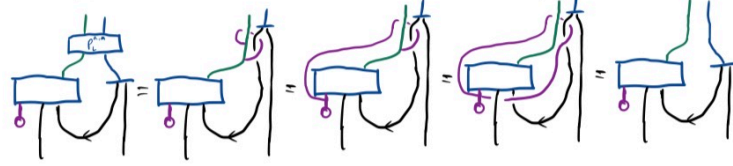
where  $\delta_{\bar{X}}: \bar{X} \rightarrow \bar{X} \otimes L$  is the canonical  $L$ -coaction on  $\bar{X}$  and  $\varepsilon: L \rightarrow \mathbb{1}$  is the counit of  $L$ .

Composing these adjunctions we get an isomorphism

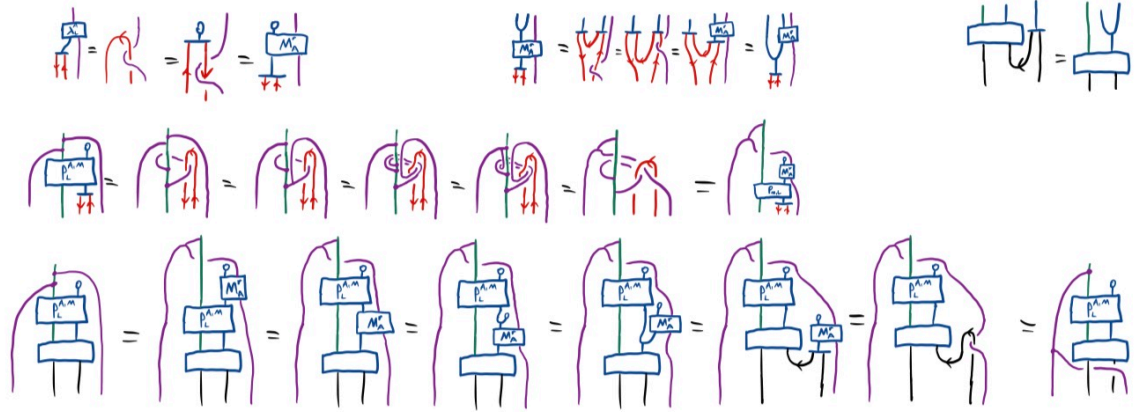
$$\mathrm{Hom}_L(X \otimes \bar{X}, \mathrm{im}(P_{M,L})) \cong \mathrm{Hom}_A(A \otimes X \otimes \bar{X}, M) \quad (5.22)$$

The claim follows from showing that the two subspaces are mapped to each other under this isomorphism. This follows from an explicit calculations using string diagrams

$$(i) \quad \mathrm{Hom}_{\mathcal{A}|_A}(A \triangleleft X \otimes \bar{X}, M) \subset \mathrm{Hom}^L(X \otimes \bar{X}, \mathrm{im}(\hat{P}_L^{A,M})):$$



$$(ii) \quad \mathrm{Hom}^L(X \otimes \bar{X}, \mathrm{im}(\hat{P}_L^{A,M})) \subset \mathrm{Hom}_{\mathcal{A}|_A}(X \otimes \bar{X} \triangleright A, M):$$



□

**Remark 5.4.** This lemma can be understood as non-semisimple variant/generalisation of [FFRS, Prop. 2.36, Prop. 3.6].

From this Lemma it follows directly that Assumption 3 is satisfied because

$$\begin{aligned} \mathrm{Hom}_L(\mathrm{im}(P_{M,L}), \mathrm{im}(P_{M,L})) &\subset \mathrm{Hom}_L(M \otimes L, M \otimes L) \cong \mathrm{Hom}_C(M \otimes L, M) \\ \mathrm{id}_{\mathrm{im}(P_{M,L})} = P_{M,L}|_{\mathrm{im}(P_{M,L})} &\mapsto P_{M,L} \qquad \qquad \qquad \mapsto (\mathrm{id}_M \otimes \lambda) \circ P_{M,L} \end{aligned}$$

and  $\hat{V}_C(S^1 \times D^2) = \lambda$  for the solid torus  $S^1 \times D^2$  we cut off before.

## 5.2 Two point correlators

We now turn to study Assumption 4. For this we have to compute the linear map assigned to the connecting bordism of the identity and compare it to the composition map

$$\mathrm{Hom}_{\mathrm{Bl}_{\mathcal{C}}(\mathfrak{C})}(\underline{X}, \mathbb{F}_{\mathfrak{C}}) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathrm{Bl}_{\mathcal{C}}(\mathfrak{C})}(\mathbb{F}_{\mathfrak{C}}, \underline{Y}) \rightarrow \mathrm{Hom}_{\mathrm{Bl}_{\mathcal{C}}(\mathfrak{C})}(\underline{X}, \underline{Y}). \quad (5.23)$$

As before we will split the discussion into a boundary and a bulk section.

### 5.2.1 Boundary

Let us start with the defect interval  $I_{n,m}^A$  from above. The connecting bordism  $M_{\mathrm{id}_{I_{n,m}^A}}^{X,Y}$ , with  $X, Y \in \mathcal{C}$ , is given by:

$$(5.24)$$

It is straightforward to see that the linear map we get from the orbifold procedure is given by

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(X, m^* \otimes n) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathcal{C}}(m^* \otimes n, Y) &\rightarrow \mathrm{Hom}_{\mathcal{C}}(X, Y) \\ f \otimes_{\mathbb{K}} g &\mapsto g \circ P_{n,m^*}^A \circ f \end{aligned} \quad (5.25)$$

with  $P_{n,m^*}^A$  the idempotent used in the verification of Assumption 3. Moreover, since  $m^* \otimes_A n \cong \mathrm{im}(P_{n,m^*}^A)$  this map restricts to the composition map on the subspace  $\mathrm{Hom}_{\mathcal{C}}(X, m^* \otimes_A n) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathcal{C}}(m^* \otimes_A n, Y)$ .

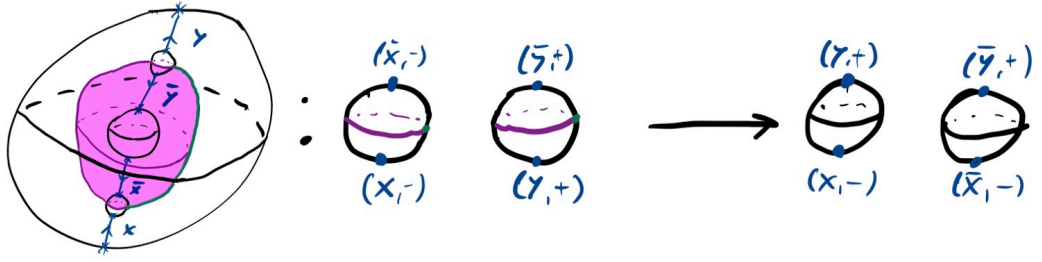
### 5.2.2 Bulk

Let us now come to the defect circle  $S_{A,M}^1$  from above. For  $(X, \overline{X}), (Y, \overline{Y}) \in \mathcal{C} \times \overline{\mathcal{C}}$  the connecting bordism

$$M_{\mathrm{id}_{S_{A,M}^1}}^{(X, \overline{X}), (Y, \overline{Y})} : M_{S_{A,M}^1}^{(X, \overline{X})} \sqcup M_{S_{A,M}^1}^{(Y, \overline{Y})} \rightarrow \widehat{\mathrm{id}_{S_{A,M}^1}}^{(X, \overline{X}), (Y, \overline{Y})} \quad (5.26)$$

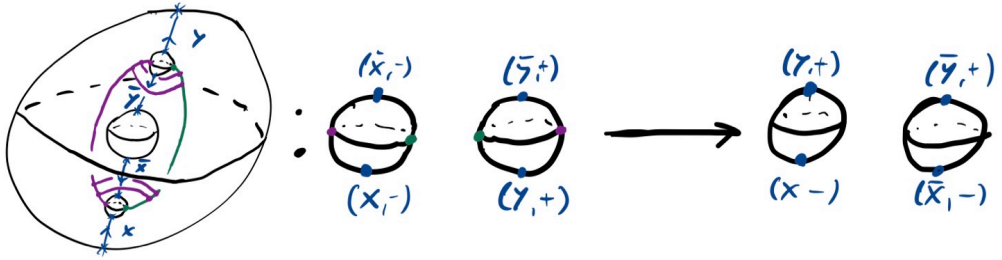
of  $\mathrm{id}_{S_{A,M}^1}$  is given by a bordism which has as underlying 3-manifold  $S^3$  with two incoming and two outgoing 3-balls cut out and  $\mathrm{id}_{S_{A,M}^1}$  surface defect connecting the

two incoming spherical boundary components:



(5.27)

Under the orbifold procedure this becomes the following morphism in  $\widehat{\text{Bord}}_{3,2}^{\chi}(\mathcal{C})$ :



(5.28)

To continue we decompose this bordism as follows:

(5.29)

From this decomposition we see that the linear map obtained by applying the TFT is a composition of two projections analogous to (5.10) and (a variant of) the linear map [HR2, Eq. (6.15)] used in the verification of this assumption for the transparent surface defect. From these observations an analogous argument as in [HR2, Sec. 6.2.2] shows that the natural transformation induced by  $Z_C(M_{\text{id}_{S_k^1}}^{(X, \bar{X}, Y, \bar{Y})})$  is indeed the composition map.

### 5.3 Algebraic structure on the field objects

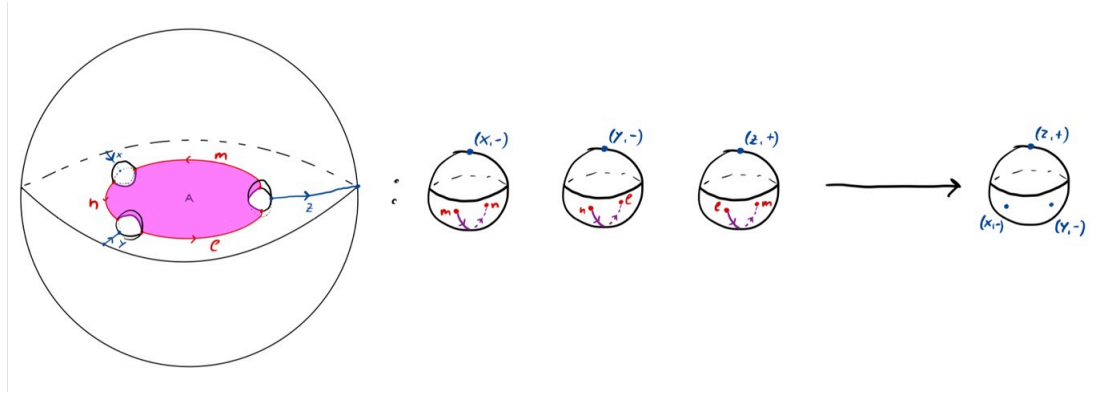
We now want to study the algebraic structures on the objects of bulk and boundary fields induced by the correlator construction. More specifically, we will show that the correlators of genus zero world sheets with at most three boundary components induce the structure of a symmetric Frobenius algebra structure on the field objects.<sup>6</sup> Moreover, we will show that these coincide with the ones obtained from the description of internal homs of pivotal module categories and thus confirm the pro-

<sup>6</sup>More precisely, we will find Frobenius algebroid structures.

posal of [FS3]. Finally, we will discuss how the object of bulk and boundary fields, for a fixed boundary condition, naturally obtain the structure of a (non-semisimple generalisation) Cardy algebra .

### 5.3.1 Boundary Frobenius algebra

We begin with the open sector. Let  $m, n, l \in {}_A\mathcal{C}$  and consider  $\mathfrak{M}_{l,n,m}^\partial: I_{l,n} \otimes I_{n,m} \rightarrow I_{l,m}$  the world sheet with underlying bordism the flat pair of pants and trivial stratification with the unique 2-stratum labelled with  $A$ . For  $X, Y, Z \in \mathcal{C}$  the connecting manifold of  $\mathfrak{M}_{l,n,m}^\partial$  is given by

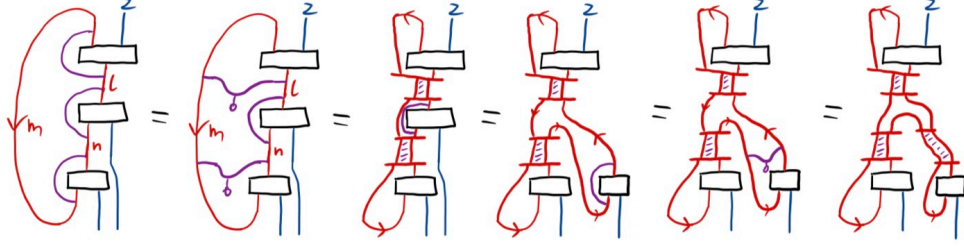


Applying  $Z_C$  and identifying the state spaces with hom spaces as before we obtain (after some simplifications) the linear map

$$\begin{aligned} \text{Hom}_A(m \otimes X, n) \otimes_{\mathbb{k}} \text{Hom}_A(n \otimes Y, l) \otimes_{\mathbb{k}} \text{Hom}_A(l, m \otimes Z) &\rightarrow \text{Hom}_{\mathcal{C}}(X \otimes Y, Z) \\ f \otimes_{\mathbb{k}} g \otimes_{\mathbb{k}} h &\mapsto \text{ptr}_m(h \circ g \circ (f \otimes \text{id}_Y)) \end{aligned} \quad (5.30)$$

where  $\text{ptr}_m$  stands for the (left) partial trace over  $m$ . We now want to compute the unique morphism  $\underline{\text{Hom}}_A(m, n) \otimes \underline{\text{Hom}}_A(n, l) \rightarrow \underline{\text{Hom}}_A(m, l)$  induced by the Yoneda lemma. To this end it is useful to first re-express the image of the top horizontal

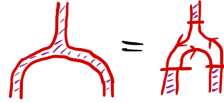
map as follows:



(5.31)

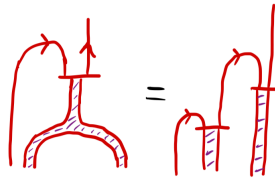
where we identified the internal homs  $\underline{\text{Hom}}_A(-, -)$ , depicted as red strands with violet core, with the image of the idempotents  $P_{-, -}^A$  discussed in the discussion of Assumption 3.

We claim that the morphism  $\underline{\text{Hom}}_A(m, n) \otimes \underline{\text{Hom}}_A(n, l) \rightarrow \underline{\text{Hom}}_A(m, l)$  in the center is the internal composition morphism of inner Homs



(5.32)

To see this recall, eg from [EGNO, Sec. 7.9], that the internal composition is defined as the unique morphism satisfying



(5.33)

A direct computation now shows that the morphism indeed satisfies this property:

(5.34)

where in the third step we used that the inclusions  $\underline{\text{Hom}}_A(-, -) \rightarrow (-)^* \otimes -$  are  $A$ -cobalanced and in the last step  $\Delta$ -separability of  $A$ .

Analogous arguments show that the correlators of the flat cup  $\emptyset \rightarrow I_{m,m}$ , the flat co-pair-of-pants  $I_{m,l} \rightarrow I_{m,n} \sqcup I_{n,l}$  and the flat cap  $I_{m,m} \rightarrow \emptyset$  coincide precisely with the symmetric Frobenius algebroid structure on  $\underline{\text{coHom}}_A(-, -) \cong \underline{\text{Hom}}_A(-, -)$  coming from the pivotal module category  ${}_A\mathcal{C}$ . With this we have the main result of this subsection.

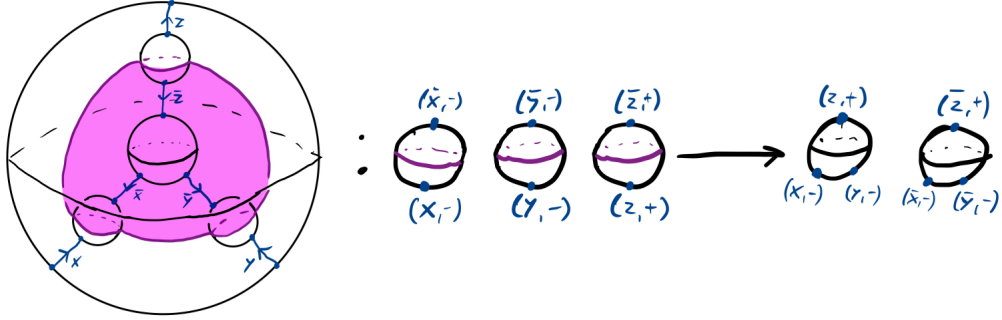
**Proposition 5.5.** The objects of boundary fields  $\mathbb{F}_{I_{m,n}^A}$  together with the boundary genus zero correlators form a symmetric Frobenius algebroid in  $\mathcal{C}$ . In particular  $\mathbb{F}_{I_{m,m}}$  is a symmetric Frobenius algebra Morita equivalent to  $A$ .

### 5.3.2 Bulk Frobenius algebra

Next we concentrate on the Frobenius algebra structure of the bulk fields. In principle we could skip this and instead discuss the defect/twisted sector directly as the bulk sector can be obtained by considering only transparent defects. However, we still choose to include a discussion of the bulk sector here explicitly for the following reasons. First of all it illustrates the general strategy for computations involving internal natural transformations in a cleaner way because the notation is less cluttered. Also from a physical perspective the bulk sector deserves special attention as it is the one which is relevant in most applications. Finally, in order to talk about the relation between bulk and boundary algebras it will be convenient to refer to this discussion.

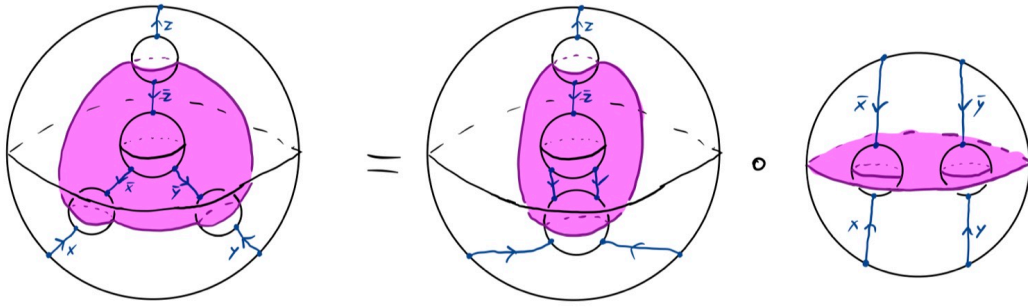
Now let  $A$  be a special, symmetric Frobenius algebra in  $\mathcal{C}$  and let  $\mathfrak{M}_A$  be the

A-labelled pair of pants world sheet. The connecting bordism of  $\mathfrak{M}_A$  is given by



(5.35)

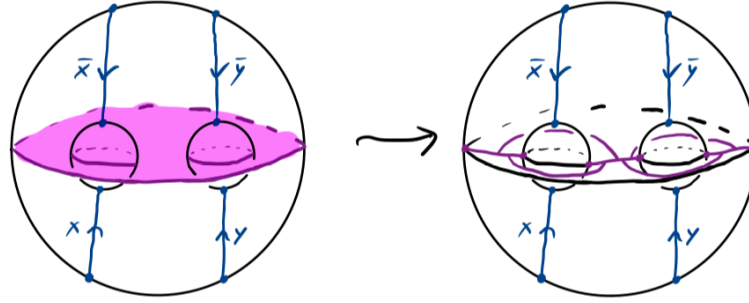
To compute  $\text{Cor}_{\mathfrak{M}_A}$  it is convenient to decompose the connecting bordism as follows



(5.36)

Now first note that the first bordism is almost the same as the one considered in the computation of the bulk two point correlator in Section 5.2. The only difference is that the bordism here has two more line defects. However, under evaluation with the defect TFT this is the same as just one line defect labelled with the product of the two labels. Thus this bordism simply induces the composition map.

Now for the second bordism the orbifold construction leads us to study



(5.37)

After evaluation and identification of the TFT state spaces with Hom spaces we arrive at the linear map

$$\begin{array}{ccccc}
 \text{Hom}_{\mathcal{C}}(X \otimes A \otimes \bar{X}, A) \otimes_{\mathbb{K}} \text{Hom}_{\mathcal{C}}(Y \otimes A \otimes \bar{Y}, A) & \xrightarrow{\quad} & \text{Hom}_{\mathcal{C}}(X \otimes Y \otimes A \otimes \bar{X} \otimes \bar{Y}, A) \\
 \downarrow & \begin{array}{c} \text{[Diagram: box with two inputs and one output]} \\ \downarrow \\ \text{[Diagram: box with two inputs and one output]} \end{array} & \xrightarrow{\quad} & \begin{array}{c} \text{[Diagram: box with two inputs and one output]} \\ \downarrow \\ \text{[Diagram: box with two inputs and one output]} \\ \downarrow \\ \text{[Diagram: box with two inputs and one output]} \end{array} & \downarrow \\
 \text{Hom}_{A|A}(X \otimes^+ A \otimes \bar{X}, A) \otimes_{\mathbb{K}} \text{Hom}_{A|A}(Y \otimes^+ A \otimes \bar{Y}, A) & \xrightarrow{\quad} & \text{Hom}_{A|A}(X \otimes Y \otimes^+ A \otimes \bar{X} \otimes \bar{Y}, A)
 \end{array}$$

(5.38)

The induced morphism  $\underline{\mu}: Z(A) \otimes Z(A) \rightarrow Z(A)$  is defined, via the Yoneda lemma,

such that the following diagram commutes

$$\begin{array}{ccccc}
 \text{Hom}_{A|A}(X \otimes_A^+ A \otimes \bar{X}, A) & \otimes_K & \text{Hom}_{A|A}(Y \otimes_A^+ A \otimes \bar{Y}, A) & \longrightarrow & \text{Hom}_{A|A}((X \otimes Y) \otimes_A^+ A \otimes (\bar{X} \otimes \bar{Y}), A) \\
 \cong \downarrow & & \downarrow & \longrightarrow & \downarrow \cong \\
 \text{Hom}_L(X \otimes \bar{X}, Z(A)) & \otimes_K & \text{Hom}_L(Y \otimes \bar{Y}, Z(A)) & \longrightarrow & \text{Hom}_L((X \otimes Y) \otimes (\bar{X} \otimes \bar{Y}), Z(A))
 \end{array}
 \quad (5.39)$$

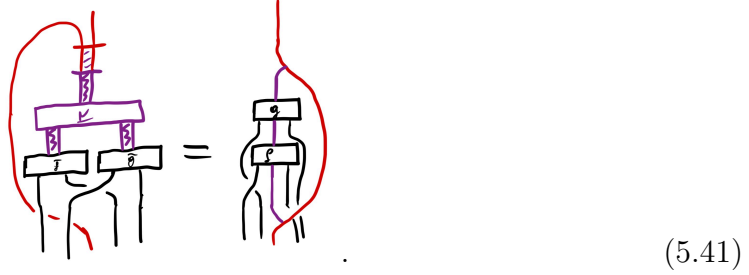
where the vertical isomorphisms come from the adjunction defining the inner Nat, depicted as a thick violet ribbon.

Now in order to obtain a concrete form of  $\underline{\mu}$  we need to understand the adjunction defining the inner Nat better. To this end recall from [FS2, Thm. 4.19] that the inner Nat can be realised as the end of the inner Hom, ie  $\text{Nat}(F, G) \cong \int_{m \in A^C} \underline{\text{Hom}}_A(F(m), G(m))$ , and analogously for the co-versions. In particular, in our setting the commutative diagram [FS2, Eq. (4.18)] becomes

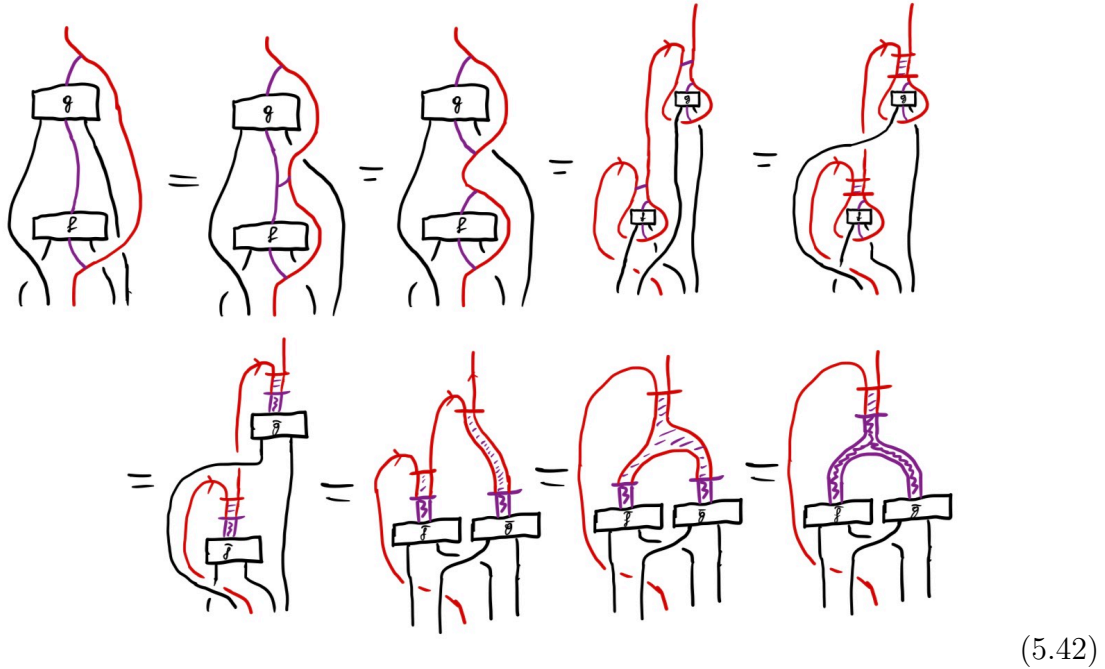
$$\begin{array}{ccc}
 \text{Hom}_{A|A}(Y \otimes_A^+ A \otimes \bar{Y}, A) & \xrightarrow{\cong} & \text{Hom}_L(Y \otimes \bar{Y}, Z(A)) \\
 \downarrow & \longrightarrow & \downarrow \\
 \text{Hom}_A(Y \otimes^+ n \otimes \bar{Y}, n) & \xrightarrow{\cong} & \text{Hom}_e(Y \otimes \bar{Y}, \underline{\text{Hom}}_A(n, n))
 \end{array}
 \quad (5.40)$$

where the horizontal maps are the adjunction isomorphisms and the vertical map on

the right the postcomposition with the universal dinatural transformation from the end formula of  $Z(A)$ . Combining this with diagram (5.39) we find that  $\underline{\mu}: Z(A) \otimes Z(A) \rightarrow Z(A)$  is fixed by satisfying



With this we can compute it now directly as follows:



In the first step we used the  $\Delta$ -separability and the module morphism property of  $f$  (or  $g$ ). In the second step we use the (co)module structure on the red ribbon while the third step uses duality. In the fourth step we use that the idempotent  $P_{n^*,n}^A: n^* \otimes n \rightarrow n^* \otimes n$  has  $\underline{\text{Hom}}_A(n, n)$  as image. In the sixth step and seventh step we use isotopy invariance as well as the definition of the multiplication on  $\underline{\text{Hom}}_A(n, n)$ . Finally, in the last step we use that the universal dinatural transformation  $Z(A) \rightarrow \underline{\text{Hom}}_A(n, n)$

is an algebra morphism by [FS2, Prop. 11]. Thus we find that  $\underline{\mu}: Z(A) \otimes Z(A) \rightarrow Z(A)$  is indeed the natural multiplication on  $Z(A)$ .

By an analogous argument one shows that the  $A$ -labelled cup world sheet  $\emptyset \rightarrow S_A^1$  induces the unit for this multiplication. Furthermore, the Frobenius comultiplication and counit on  $Z(A)$  can also be obtained as the correlators for the co-pair-of-pants and the cap by using the coend structure of  $Z(A)$  instead. In summary we have:

**Proposition 5.6.** The object of bulk fields  $\mathbb{F}_{S_A^1} \in \mathcal{C} \boxtimes \overline{\mathcal{C}}$  together with the bulk genus zero correlators is a commutative symmetric Frobenius algebra which is isomorphic to the full center  $Z(A)$ .

Moreover, the mapping class group invariance of  $\text{Cor}_-$  implies:

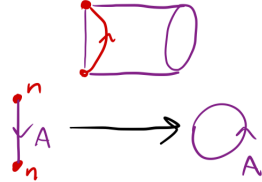
**Corollary 5.7.** The commutative Frobenius algebra  $Z(A)$  is modular in the sense of [FS1, Def. 4.9].

### 5.3.3 Defect Frobenius algebroid

TODO:

### 5.3.4 Bulk-Boundary morphisms and Cardy algebras

In this section we will study how the object of boundary and bulk fields are related. To this end we fix  $n \in {}_A\mathcal{C}$  and consider the whistle world sheet  $\mathfrak{W}_{A,n}: I_{n,n} \rightarrow S_A^1$  given by



(5.43)

For  $X, Y, \bar{Y} \in \mathcal{C}$  the connecting bordism of  $\mathfrak{W}_{A,n}$  is given by

(5.44)

After applying  $Z_C$  and the usual identifications as well as simplifications we get the linear map

$$\text{Hom}_A(X \otimes n, n) \otimes_{\mathbb{K}} \text{Hom}_{A^{\text{op}}} (A, Y \otimes A \otimes \bar{Y}) \longrightarrow \text{Hom}_E(X, Y \otimes \bar{Y})$$

(5.45)

Using the coend version of Diagram 5.40, a direct computation now shows that the induced morphism  $\mathbb{F}_{I_{n,n}} \rightarrow \mathbb{F}_{S_A^1}$  coincides with the universal dinatural transformation  $\iota_n^A: \underline{\text{Hom}}_A(n, n) \rightarrow Z(A)$  coming from the coend description of  $Z(A)$ :

(5.46)

where in the first step  $\Delta$ -separability and the module morphism structure of  $g$  and in the last step commutativity of the coend version of Diagram 5.40 was used.

Analogously the adjoint whistle world sheet  $S_A^1 \rightarrow I_{n,n}$  induces the universal dinatural transformation  $\pi_n^A: Z(A) \rightarrow \underline{\text{Hom}}_A(n, n)$  from the end description of  $Z(A)$ .

**Definition 5.8.** A *Cardy pair* of  $\mathcal{C}$  is a triple  $(A_{\text{op}}, A_{\text{cl}}, i_{\text{cl-op}})$  where  $(A_{\text{op}}, \mu_{\text{op}}, \eta_{\text{op}}, \Delta_{\text{op}}, \varepsilon_{\text{op}})$  is a symmetric Frobenius algebra in  $\mathcal{C}$ ,  $(A_{\text{cl}}, \mu_{\text{cl}}, \eta_{\text{cl}}, \Delta_{\text{cl}}, \varepsilon_{\text{cl}})$  is a modular Frobenius algebra in  $\mathcal{Z}(\mathcal{C})$ , and  $i_{\text{cl-op}}: A_{\text{cl}} \rightarrow A_{\text{op}}$  is an algebra morphism in  $\mathcal{C}$ , such that the following conditions are satisfied:

(i) Center condition: The diagram

commutes where  $\gamma_{A_{\text{op}}}^{A_{\text{cl}}}$  is the  $A_{\text{op}}$  component of the half brading of  $A_{\text{cl}}$ .

(ii) Cardy condition:

## 5.4 Further CFT quantities

### 5.4.1 Boundary states

Boundary states are equivalent to bulk one point correlators on a disk with fixed boundary condition. In our formulation these correspond to correlators for a world sheet  $\mathfrak{D}$  with underlying surface a cylinder such that one boundary is a gluing boundary and the other one a free boundary. The 2-stratum will be labelled with a fixed special symmetric Frobenius algebra  $A \in \mathcal{C}$ , and the free boundary with a left  $A$ -module  $n \in {}_A\mathcal{C}$ . In general one needs to distinguish between incoming and outgoing boundary states. For brevity we will only briefly discuss the case of outgoing boundary states to illustrate the differences to the case of trivial surface defect treated in [HR2, Sec.6.3.1].

We consider the world sheet  $\mathfrak{D}_{A,n}: \emptyset \rightarrow S_A^1$  where  $S_A^1$  denotes the  $A$ -labelled circle. Let  $(X, \bar{X}) \in \mathcal{C} \times \bar{\mathcal{C}}$ , the connecting bordism  $M_{D_{A,n}}^{(X, \bar{X})}: M_{S_A^1}^{(X, \bar{X})} \rightarrow \hat{D}_{A,n}^{(X, \bar{X})}$  of  $\mathfrak{D}_{A,n}: \emptyset \rightarrow S_A^1$  is given by:

To get the linear map

$$Z_C \left( M_{D_{A,n}}^{(X,\bar{X})} \right) : Z_C \left( M_{S_A^1}^{(X,\bar{X})} \right) \rightarrow Z_C \left( \hat{D}_{A,n}^{(X,\bar{X})} \right) \quad (5.48)$$

we first note that

$$\begin{aligned} Z_C(\hat{D}_{A,n}^{(X,\bar{X})}) &= \hat{V}_C(\hat{\mathfrak{D}}_{A,n}^{(X,\bar{X})}) \\ &\cong \text{Hom}_C(\mathbb{1}, X \otimes \bar{X}) \end{aligned} \quad (5.49)$$

as  $\hat{D}_{A,n}^{(X,\bar{X})} \in \widehat{\text{Bord}}_{3,2}^X(\mathcal{C})$ . Now according the orbifold construction we have that  $Z_C \left( M_{D_{A,n}}^{(X,\bar{X})} \right)$  is given by the restriction of the linear map

$$\text{Hom}_C(A, X \otimes A \otimes \bar{X}) \rightarrow \text{Hom}_C(\mathbb{1}, X \otimes \bar{X}) \quad (5.50)$$

to  $Z_C \left( M_{S_A^1}^{(X,\bar{X})} \right) \cong \text{Hom}_{A|A}(A, X \otimes^+ A \otimes^- \bar{X})$ . On  $\text{Hom}_{A|A}(A, X \otimes^+ A \otimes^- \bar{X})$  it is straightforward to further simplify this map using the Frobenius relations and  $\Delta$ -separability leading to:

$$Z_C \left( M_{D_{A,n}}^{(X,\bar{X})} \right) : \text{Hom}_{A|A}(A, X \otimes^+ A \otimes^- \bar{X}) \rightarrow \text{Hom}_C(\mathbb{1}, X \otimes \bar{X}) \quad (5.51)$$

By definition  $\text{Cor}_{\mathfrak{D}_{A,n}} \in \text{Nat}(\text{Cor}_{S_A^1}, \text{Bl}_C(\mathfrak{D}_{A,n}))$  is now induced by the natural transformation with components  $Z_C \left( M_{D_{A,n}}^{(X,\bar{X})} \right)$ .

To get a more concrete description of  $\text{Cor}_{\mathfrak{D}_{A,n}}$  similarly to the one derived in [HR2, Sec. 6.3.1] we want to study the image of  $\text{Cor}_{\mathfrak{D}_{A,n}}$  under the chain of isomorphisms

$$\begin{aligned} \text{Nat}(\text{Cor}_{S_A^1}, \text{Bl}_C(\mathfrak{D}_n^{\text{out}})) &\cong \text{Nat}(\text{Hom}_L(\mathbb{F}_{S_A^1}, -), \text{Hom}_L(L, -)) \\ &\cong \text{Hom}_L(L, \mathbb{F}_{S_A^1}) \\ &\cong \text{Hom}_C(\mathbb{1}, \mathbb{F}_{S_A^1}) \end{aligned} \quad (5.52)$$

where we used  $\text{Cor}_{S_A^1} \cong \text{Hom}_L(\mathbb{F}_{S_A^1}, -)$ ,  $\text{Bl}_C(\mathfrak{D}_n^{\text{out}}) \cong \text{Hom}_L(L, -)$ , the Yoneda lemma, and the free-forgetful adjunction.

In contrast to [HR2, Sec. 6.3.1]. we unfortunately do not have such a concrete description of  $\text{Cor}_{\mathfrak{D}_{A,n}}$  as an element in  $\text{Hom}_C(\mathbb{1}, \mathbb{F}_{S_A^1})$  yet. However, there is a natural candidate in the form of the *internal cocharacter* generalising the cocharacter we encountered in [HR2, Sec. 6.3.1]. see [Shi3, Sec. 5.3] for the dual notion of relative cocharacters. The internal cocharacter is obtained as the composition  $\mathbb{1} \rightarrow \underline{\text{Hom}}_A(n, n) \rightarrow \int^{m \in {}_A\mathcal{C}} \underline{\text{Hom}}_A(m, m) \cong \mathbb{F}_{S_A^1}$  where the first morphism is the unit of  $\underline{\text{Hom}}_A(n, n)$  as an algebra, and the second morphism the universal dinatural transformation of the coend. Making this idea more precise is unfortunately out of scope for this outlook as we would first need to make the description of  $\mathbb{F}_{S_A^1}$  as a coend over the internal Hom stated above explicit [FS2, Thm. 18] and then use this and the natural Frobenius algebra structure on it to argue as in [HR2, Sec. 6.3.1].

### 5.4.2 Partition functions

We now come to the partition functions of the full CFTs in the form of annulus and torus correlators. Let us denote with  $\mathfrak{A}_{n,m}^A$  the annulus world sheet with boundary conditions  $m, n \in {}_A\mathcal{C}$ . The connecting bordism of  $\mathfrak{A}_{n,m}^A$  is given by

$$M_{\mathfrak{A}_{m,n}^A} = \left( \text{Diagram} \right) : \emptyset \longrightarrow \left( \text{Diagram} \right) \quad (5.53)$$

Going through the orbifold construction and employing [HR1, Eq. 4.26] for  $X = m^* \otimes n$  it is straightforward to compute that  $\text{Cor}_{\mathfrak{A}_{n,m}^A} = Z_C(M_{\mathfrak{A}_{m,n}^A}) \in \widehat{V}_C(T^2)$  corresponds to

$$\iota_{m^* \otimes n} \circ (\text{id}_{(m^* \otimes n)^*} \otimes P_{n,m^*}^A) \circ \widetilde{\text{coev}}_{m^* \otimes n} \in \text{Hom}_C(\mathbb{1}, L) \quad (5.54)$$

under the isomorphisms  $\widehat{V}_C(T^2) \cong \text{Hom}_C(L, \mathbb{1}) \cong \text{Hom}_C(\mathbb{1}, L)$  where we used the Frobenius algebra structure on  $L$  for the second one. Next recall that  $P_{n,m^*}^A$  is a split

idempotent with factorisation

$$m^* \otimes n \xrightarrow{p_{n,m^*}^A} m^* \otimes_A n \xleftarrow{i_{n,m^*}^A} m^* \otimes n. \quad (5.55)$$

Factoring  $P_{n,m^*}^A$  as above and using dinaturality as well as  $p_{n,m^*}^A \circ i_{n,m^*}^A = \text{id}_{m^* \otimes_A n}$  finally leads to

$$\text{Cor}_{\mathfrak{A}_{n,m}^A} = \iota_{m^* \otimes_A n} \circ \widetilde{\text{coev}}_{m^* \otimes_A n} = \check{\chi}_{m^* \otimes_A n}. \quad (5.56)$$

Now for the torus partition function we have to compute the correlator of the  $A$ -labelled torus  $T_A^2$ . For this we can use the same reasoning as for the Cardy case computation in [HR2, Sec. 6.3.3]. More specifically, the argument in [HR2, Sec. 6.3.3] only requires knowledge of the object of bulk fields as well as the bulk two-point correlators, which is formally just the identity on the bulk object by Assumption 4. This means we can simply substitute  $Z(A) = \mathbb{F}_{S_A^1}$  for  $\mathbb{F}_{S_{\mathbb{I}}^1} = \mathbb{L}$  in all computations. Doing this we find  $\text{Cor}_{T_A^2} = \check{\chi}_{Z(A)} \in \text{Hom}_{\mathbb{L}}(\mathbb{1}, \mathbb{L})$ . To connect with the semisimple setting we use [Shi1, Cor. 4.2] to express this in terms of characters of simple objects in  $\mathcal{C}$  as follows

$$\check{\chi}_{Z(A)} = \sum_{U,V \in \text{Irr}(\mathcal{C})} Z_{U,V}^A \check{\chi}_U \boxtimes \check{\chi}_V \quad (5.57)$$

with  $Z_{U,V}^A = [Z(A) : U \boxtimes V]$  the multiplicity of the simple  $U \boxtimes V \in \mathcal{C} \boxtimes \bar{\mathcal{C}}$  in  $Z(A)$ . Using the adjunction defining  $Z(A)$  we get

$$\begin{aligned} [Z(A) : U \boxtimes V] &= \dim_{\mathbb{k}} \text{Hom}_{\mathcal{C} \boxtimes \bar{\mathcal{C}}}(P_U \boxtimes P_V, Z(A)) \\ &= \dim_{\mathbb{k}} \text{Hom}_{A,A}(P_U \otimes^+ A \otimes^- P_V, A) \end{aligned} \quad (5.58)$$

where  $P_S$  denotes the projective cover of the simple  $S$ .

Recently, Woike [Woi1] also gave a construction of full CFT correlators in the finite non-semisimple building on the modular microcosm principle [Woi2] and on factorisation homology. There, the torus partition function is extracted [Woi1, Def. 7.2] as an endomorphism  $Z^A$  of the space of chiral conformal blocks of the torus.<sup>7</sup> For this endomorphism he then extracts matrix coefficients  $Z_{P,Q}^A$ , for  $P, Q$  projective, via the Hattori–Stallings trace on the factorisation homology of  $S^2 \times S^1$ , and shows that they are non-negative integers, bounded above by  $\dim_{\mathbb{k}}(\text{Hom}_{\mathcal{C}}(P^\vee, F \otimes Q))$  [Woi1, Thm. 7.6]. Moreover, for the Cardy case  $F = \mathbb{1}$ , he shows that this bound is saturated and  $Z_{P,Q}^1$  is the Cartan matrix entry  $\dim_{\mathbb{k}}(\text{Hom}_{\mathcal{C}}(P^\vee, Q))$  of  $\mathcal{C}$ .

Setting  $A = \mathbb{1}$  in our formula for  $[Z(A) : U \boxtimes V]$  also recovers the entries of the Cartan matrix of  $\mathcal{C}$ , so the two independently constructed torus partition functions

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<sup>7</sup>Beware of the different notation: our  $\mathcal{C}$  is denoted by  $\mathcal{A}$ , and  $A$  by  $F$ .

agree in the Cardy case. Beyond the Cardy case, however, the two results are of a somewhat different nature: our formula  $[Z(A) : U \boxtimes V] = \dim_{\mathbb{k}} \text{Hom}_{A,A}(P_U \otimes^+ A \otimes^- P_V, A)$  is an *exact* Hom-dimension in the category of  $A$ - $A$ -bimodules for every special symmetric Frobenius algebra  $A$ , from which non-negativity and integrality of  $Z_{U,V}^A$  are immediate, whereas [Woi1, Thm. 7.6] only bounds  $Z_{P,Q}^A$  from above for general  $A$ .

### 5.4.3 Algebra of defect operators

Finally, let us discuss the action of line defects on the object of bulk fields. For this let  $D \in {}_A\mathcal{C}_A$  and consider the world sheet with underlying surface the cylinder  $S^1 \times I$ , an  $M$ -labelled 1-stratum at  $S^1 \times \{1/2\}$ , and the two 2-strata labelled with  $A$ :

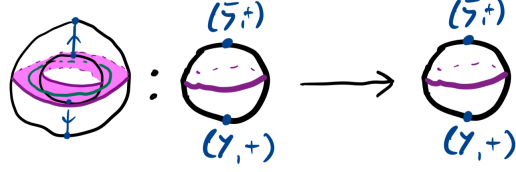
$$\mathfrak{D}_D = \text{cylinder with } D \text{ and } A \text{ strata} : \text{circle with } A \rightarrow \text{circle with } A. \quad (5.59)$$

The connecting bordism  $M_{\mathfrak{D}_D}^{(X,\bar{X},Y,\bar{Y})}$  for this world sheet is given by

$$\text{Bordism diagram with spheres and labels } (X,-), (\bar{X},-), (Y,+), (\bar{Y},+) : \text{Diagram} \rightarrow \text{Diagram}. \quad (5.60)$$

**TODO:** modify this tikz appropriately

To compute  $Z_{\mathcal{C}}(M_{\mathfrak{D}_D}^{(X,\bar{X},Y,\bar{Y})})$  we decompose it in a similar way as in Section 5.3.2, see also [HR2, Sec. 6.3.4]. This leads us to study the image of the following defect bordism



Going through the orbifold construction and identifying with Hom spaces this leads us to the linear map

$$Z_C \left( M_{\mathfrak{D}_D}^{(X, \bar{X})} \right) : \text{Hom}_{A|A}(A, Y \otimes^+ A \otimes^- \bar{Y}) \rightarrow \text{Hom}_{A|A}(A, Y \otimes^+ A \otimes^- \bar{Y})$$

(5.61)

Let us denote the natural transformation induced by this map with  $\Phi_D$ . The natural transformation induced by  $\hat{V}_C \left( M_{\mathfrak{D}_D}^{(X, \bar{X}, Y, \bar{Y})} \right)$  is the composition

$$\text{Cor}_{\mathfrak{D}_D} = \varepsilon_{\text{Cor}_{S_A^1}} \circ (\text{id}_{\text{Cor}_{S^1}^\dagger} \otimes_{\mathbb{k}} \Phi_D) : \text{Cor}_{S_A^1}^\dagger \otimes_{\mathbb{k}} \text{Cor}_{S_A^1} \Rightarrow \text{Bl}_C(\mathfrak{D}_D). \quad (5.62)$$

Next note that by the Yoneda lemma we have  $\text{Nat}(\text{Cor}_{S_A^1}^\dagger \otimes_{\mathbb{k}} \text{Cor}_{S_A^1}, \text{Bl}_C(\mathfrak{D}_D)) \cong \text{Hom}_L(\mathbb{F}_{S_A^1}, \mathbb{F}_{S_A^1})$ , so instead of computing the corresponding natural transformation we will focus on the endomorphism of  $\mathbb{F}_{S_A^1} \cong Z(A)$  instead. We will call this endomorphism the *defect operator* associated to  $D$  and denote it with  $\mathcal{O}_D$ . Since  $\varepsilon_{\text{Cor}_{S_A^1}}$  corresponds to the composition map by the results of Section 5.2 it we find that  $\mathcal{O}_D$  is also induced by  $\Phi_D$ .

To continue first recall that  $Z(A) \cong \int^{m \in {}_A \mathcal{C}} \underline{\text{Hom}}_A(m, m)$  by pivotality  ${}_A \mathcal{C}$  and let us denote the corresponding universal dinatural with  $\iota_m^A : \underline{\text{Hom}}_A(m, m) \rightarrow Z(A)$ .

Next note that for any module functor  $F \in \mathcal{R}\text{ex}_C({}_A \mathcal{C}, {}_A \mathcal{C})$  we have the following

commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_A(m \triangleleft X, n) & \xrightarrow{F} & \mathrm{Hom}_A(F(m \triangleleft X), F(n)) & \xrightarrow{\cong} & \mathrm{Hom}_A(F(m) \triangleleft X, F(n)) \\
\cong \downarrow & & & & \downarrow \cong \\
\mathrm{Hom}_C(X, \underline{\mathrm{Hom}}_A(m, n)) & \xrightarrow{\hspace{10em}} & \mathrm{Hom}_C(X, \underline{\mathrm{Hom}}_A(F(m), F(n))) & & 
\end{array} \tag{5.63}$$

where the vertical isomorphism come from the adjunction defining the inner Hom and the horizontal isomorphism on the top from the module functor constraint. This diagram uniquely defines a family of morphisms

$$\underline{F}_{m,n}: \underline{\mathrm{Hom}}_A(m, n) \rightarrow \underline{\mathrm{Hom}}_A(F(m), F(n)) \tag{5.64}$$

by the Yoneda lemma. Moreover, this family is natural in  $m, m' \in {}_A\mathcal{C}$  and it is straightforward to show that it is multiplicative in the sense that  $\underline{F} \circ \underline{G} = \underline{F} \circ \underline{G}$  and  $\underline{\mathrm{id}}_{m,m} = \mathrm{id}_m$ .

The family  $i_{F(m)}^A \circ \underline{F}_{m,m}: \underline{\mathrm{Hom}}_A(m, m) \rightarrow Z(A)$  is dinatural in  $m$  by naturality of  $\underline{F}$ . Thus there exists a unique morphism  $\beta_F: Z(A) \rightarrow Z(A)$  such that

$$\beta_F \circ i_m^A = i_{F(m)}^A \circ \underline{F}_{m,m} \tag{5.65}$$

by the universal property of the coend. We claim that  $\beta_F = \mathcal{O}_D$  for  $F = D \otimes_A -$ . **TODO:** add proof of this

**Proposition 5.9.** The assignment

$$\begin{aligned}
\mathrm{Gr}({}_A\mathcal{C}_A) &\rightarrow \mathrm{End}(Z(A)) \\
[D] &\mapsto \mathcal{O}_D.
\end{aligned} \tag{5.66}$$

is a well defined ring homomorphism.

**Remark 5.10.** We want to make some comments on the relation and one key difference to the original construction of [FRSI]. First we note that if  $\mathcal{C}$  is semisimple then the above discussion with special, symmetric Frobenius algebras to label the surface defects recovers precisely the work of [FRSI]. This essentially follows by combining [DGGPR, Rem. 4.9] with [KS] and [FSV]. See also the discussion in [FS3, Sec. 3.4] for how the description of the field content in the semisimple setting is encoded by the inner homs we encountered above. It turns out, that in this setting only the *Morita classes* of these algebras are seen by the construction [FSV]. This raises the question if this is still true beyond semisimplicity. The answer to this

question is: no. In the non-semisimple setting  $\Delta$ -separability is no longer a Morita invariant notion. To see this recall, e.g. from [EGNO, Exa. 7.10.2] that the monoidal unit  $\mathbb{1} \in \mathcal{C}$  is Morita equivalent to  $X \otimes X^*$ , with algebra structure as in [EGNO, Exa. 7.8.4], for any  $X \in \mathcal{C}$ . Now note that the unit  $\mathbb{1}$  is always  $\Delta$ -separable as an algebra,  $X \otimes X^*$  on the other hand is only  $\Delta$ -separable if the categorical dimension  $\dim_{\mathcal{C}}(X)$  is non-zero. Thus, for  $X$  projective  $X \otimes X^*$  cannot be  $\Delta$ -separable.

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